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Some Properties of Unit, Zero-Divisor and Idempotent Elements of the Ring of Dual Numbers of the Ring Z_n, $Z_n[\alpha]$

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Abstract

For any commutative ring with unity R, a new ring can be defined called the ring of dual numbers of the ring R, denoted by $R[\alpha]$. In this paper we obtain a characterization of unit and zero-divisor elements in the ring of dual numbers of the ring of integers modulo n, $Z_n[\alpha]$.

Keywords: Ring of Integers modulo n, Ring of Dual Numbers, Unit Element, Zero-Divisor Element.

1. Introduction

All rings considered in this paper are commutative with unity. Unity played a fundamental role in the rings. A nonzero element u in the ring R is called a unit if there exists an element v in the ring R such that $u.v = 1_R$. In fact, this element v is unique, denoted by u^{-1}.

For any ring R, a new ring which is an extension of R can be defined, denoted by $R[\alpha]$, called the ring of dual numbers of the ring R as follows:

$R[\alpha] = \{a + b\alpha\colon a, b \in R, \alpha^2 = 0\}$.

$(a + b\alpha) + (c + d\alpha) = (a + c) + (b + d)\alpha$. $(a + b\alpha)(c + d\alpha) = ac + (ad + bc)\alpha$.

In this paper, I try to find some properties of unit, zero-divisor and idempo- tent elements in the ring of dual numbers of the ring of integers modulo n, $Z_n[\alpha]$.

Theorem 1.1: A nonzero element $a \in Z_n$ is

1. A unit element in Z_n if and only if $gcd\ (a, n) = 1$.
2. A zero-divisor element in Z_n if and only if $gcd\ (a, n) \neq 1$.

Example 1.2: In Z_6:

1. The unit elements are 1 ($mod6$) and 5($mod6$).
2. The zero-divisor elements are 2 ($mod6$), 3($mod6$) and 4($mod6$).

2. Unit Elements of $Z_n[\alpha]$

Lemma 2.1: An element $u + v\alpha$ is a unit in $Z_n[\alpha]$ iff u is a unit in Z_n.

Proof: Assume that $u + v\alpha$ is a unit in $Z_n[\alpha]$, then there exists $c + d\alpha$ in $Z_n[\alpha]$ such that $(u + v\alpha)\ (c + d\alpha) = 1$, and so $uc + (ud + vc)\ \alpha = 1$.

Hence $uc = 1$. i.e u is a unit in Z_n.

Conversely, assume that u is a unit in Z_n, then u^{-1} exists in Z_n.

Consider $(u + v\alpha)\ (u^{-1} - u^{-2}v\alpha) = uu^{-1} - u^{-1}v\alpha + u^{-1}v\alpha = uu^{-1} = 1$.

Thus, $u + v\alpha$ is a unit in $Z_n[\alpha]$.

Theorem 2.2: The set of all unit elements in $Z_n[\alpha]$ is:

$\{u + v\alpha$: u is a unit in $Z_n\}$.

Proof: See the previos lemma.

3. Zero-Divisor Elements of $Z_n[\alpha]$

Lemma 3.1: An element $x + y\alpha$ is a zero - divisor in $Z_n[\alpha]$ iff either $x = 0$ and $y \neq 0$ in Z_n or x is a zero - divisor in Z_n.

Proof: Assume that $x + y\alpha$ is a zero - divisor in $Z_n[\alpha]$, then $x + y\alpha \neq 0$.

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If $x = 0$, then $y \neq 0$ because $x + y\alpha \neq 0$.

If $x \neq 0$, then there exists $c+d\alpha \in Z_n[\alpha]-\{0\}$ such that $(x+y\alpha)(c+d\alpha) = 0$.

Hence $xc = 0$.......(1), and $xd + yc = 0$. (2).

If $c \neq 0$, then by (1) x is a zero - divisor in Z_n.

If $c = 0$, then by (2) $xd = 0$.

But $c + d\alpha \neq 0$, so $d \neq 0$. Thus x is a zero - divisor in Z_n.

Conversely, assume that $x = 0$, then $y\alpha$ is zero - divisor because $y \neq 0$ and $y\alpha^2 = 0$.

Now assume that x is a zero - divisor in Z_n. So, there exists $v \in Z_n - \{0\}$ such that $xv = 0$. (3).

Consider $(x + y\alpha)\ (v\alpha) = xv\alpha = 0$ by (3), for any $y \in Z_n$.

Thus, $x + y\alpha$ is zero - divisor in $Z_n\alpha$.

Theorem 3.2: The set of all zero - divisor elements in $Z_n[\alpha]$ is:

$\{x + y\alpha$: x is a zero - divisor in $Z_n\}$.

Proof: See the previos lemma.

4. Idempotent Elements of $Z_n[\alpha]$

Definition 4.1: An element $a \in Z_n$ is said to be idempotent element if $a^2 = a$.

Example 4.2: The idempotent elements in Z_6 are the elements:

$\{0(mod 6), 1(mod 6), 3(mod 6), 4(mod 6)\}$.

Lemma 4.3: The elements 0 and 1 are always idempotent elements in Z_n and $Z_n[\alpha]$.

Proof: Easy.

Lemma 4.4: For any nonzero element $b \in Z_n$, $b\alpha$ isnt idempotent element in $Z_n[\alpha]$.

Proof: Assume that $b\alpha$ is an idempotent element in $Z_n[\alpha]$ for any nonzero element $b \in Z_n$.

So, $(b\alpha)^2 = b^2\alpha^2 = b^2 0 = 0 \neq b\alpha$, which is a contradiction. Thus, $b\alpha$ isnt idempotent element in $Z_n[\alpha]$ for any nonzero b.

Lemma 4.5: For any nonzero element $b \in Z_n$, $1+b\alpha$ isnt idempotent element in $Z_n[\alpha]$.

Proof: Assume that $1 + b\alpha$ is an idempotent element in $Z_n[\alpha]$ for any nonzero element $b \in Z_n$.

So, $(1 + b\alpha)^2 = 1 + b\alpha$, and so $2b = b$, Thus, $b = 0$, which is a contradiction. Hence, $1 + b\alpha$ isnt idempotent element in $Z_n[\alpha]$.

Lemma 4.6: For any nonzero unit element $a \in Z_n[\alpha] - \{1\}$, a isnt idempo- tent element in $Z_n[\alpha]$.

Proof: Assume that a is an idempotent element in $Z_n[\alpha] - \{1\}$, so $a^2 = a$.

But a is a unit in Z_n, so $a^2 = a$ implies $a = 1$, which is a contradiction because $a \neq 1$.

Thus, a isnt idempotent element.

Corollary 4.7: For any nonzero unit element $a \in Z_n[\alpha] - \{1\}$, $a + b\alpha$ isnt idempotent element in $Z_n[\alpha]$ for any nonzero element b in Z_n.

Theorem 4.8: For any nonzero element $b \in Z_n$, $a + b\alpha$ isnt idempotent element in $Z_n[\alpha]$.

Proof: Assume that $a + b\alpha$ is an idempotent element in $Z_n[\alpha]$.

So, $(a + b\alpha)^2 = a + b\alpha$, and so $a^2 = a$ ....(1) and $2ab = b$ (2).

Thus, we have four cases to study:

Case 1: If $a = 0$, then by Lemma 4.4 we are done.

Case 2: If $a = 1$, then by Lemma 4.5 we are done.

Case 3: If a is a nonzero unit in $Z_n[\alpha]$ such that $a \neq 1$, then by Corollary 4.7 we are done.

Case 4: If a is a nonzero nonunit in $Z_n[\alpha]$, then by (2) we have $(2a-1)b = 0$, and so we have:

If $2a - 1 = 0$, then a is a unit, which is a contradiction.

If $b = 0$, we have a contradiction because $b \neq 0$.

If neither $2a - 1 = 0$ nor $b = 0$, then $Z_n[\alpha]$ is an integral domain, which is a contradiction.

Consequently, $a + b\alpha$ isnt idempotent element in $Z_n[\alpha]$ for any nonzero element b.

Corollary 4.9: If $a + b\alpha \in Z_n[\alpha]$ is an idempotent element then a is an idempotent element in Z_n.

Proof: Assume that $a + b\alpha$ is an idempotent element in $Z_n[\alpha]$.

So, $(a + b\alpha)^2 = a + b\alpha$, and so $a^2 = a$ ....(1) and $2ab = b$ (2).

By (1), a is an idempotent element in Z_n.

The converse of the previos corollary isnt true, the next example will explain that.

Example 4.10: In Z_6, 3 is an idempotent element but $3 + \alpha$ isnt idempotent element in $Z_6[\alpha]$.

Theorem 4.11: The characterization of idempotent element in $Z_n[\alpha]$ have the same characterization in Z_n, see [5].

References

1. D. D. Anderson and Sangmin Chun, Irreducible Elements in Rings with Zero Divisors, *Housten Journal of Mathematics*, **37**, (2011) 741- 744.
2. D. D. Anderson and Silvia Voldes-Leon, Factorization in Commutative Rings with Zero Divisors, *Rocky Mountain Journal of Mathmatics* **26** (1996) 439-479.
3. Adam A. Allan, Micheal J. Dunne, John R. Jack, Justin C. Lynd and Harold W.Ellingsen Jr, Classification of the group of units in Gaussian integers modulo n,*Mu Epsilon Journal* **12 (9)** (Fall 2008), 513-519.
4. S. Galovich, Unique Factorization Rings With Zero Divisors, *Math Magazine* **51**(1978) 276-283.
5. Suman Mondal and Dhiren K Basnet, Idempotent of Z_n, arxiv 2410.14576v1.
6. Thomas Q Sibley. Idempotents à la mod. The College Mathematics Journal, 43(5):401–404, 2012.
7. Gaurav Mittal. Non trivial idempotents of the matrix rings over poly- nomial ring Zpqr[x].preprint.