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# On Fine Rings and Their Relationship with Clean, Nil-Clean and Polynomial Ring Extensions

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## Abstract

The structural characteristics of fine rings and their connections to polynomial rings, clean rings, and nil-clean rings are examined in this study. One important class that is closely connected to simple rings is fine rings, which are defined via decompositions involving units and nilpotent elements. We examine whether the fine property is preserved under polynomial extensions and show that, in general, polynomial rings fail to retain this property. Furthermore, we analyze the connections between fine, clean, and nil-clean rings, highlighting that while every fine is structurally rigid and linked to simplicity, clean and nil-clean rings exhibit more flexible behavior, especially in the presence of nilpotent elements. In particular, we study rings of the form $F_q[x]/(x^n)$, establishing conditions under which they are clean or nil-clean, and explaining why they are not fine. The results provide a clearer structural comparison between these classes and suggest directions for further research in decomposition theory and ring extensions.

**Keywords:** Fine, Clean, Nil-Clean and Polynomial Ring.

## 1. Introduction

Ring theory plays a fundamental role in modern algebra, with various classes of rings studied extensively due to their rich structural properties and wide-ranging applications. Among these classes, fine rings have recently attracted attention because of their unique decomposition properties and their relationships with other significant ring classes. Understanding how fine rings interact with well-studied structures such as polynomial rings and clean rings offers insights into their internal algebraic behavior and potential applications.

Polynomial rings are central objects in algebra, providing a bridge between ring theory and algebraic geometry. On the other hand, clean rings, characterized by the decomposition of elements into sums of units and idempotents, present a framework for analyzing ring elements in terms of their simpler components. Furthermore, by permitting decomposition into sums using multiple units and idempotents, the notion of nil-clean rings generalizes clean rings. If ∃ a ∈ R and $a = u + t$, so as to u is unit in R and t is nilpotent in R then, we get a nonzero ring called fine ring. when R=0, we get a trivial case, where $u + t = 0 \rightarrow u = -t$.

This study focuses on the three sets Idem(R), Nil(R), and U(R) in every ring R. They represent the idempotent, nilpotent and unit elements in R, respectively [4]. In addition to that explore the structural relationships and characterizations of fine rings in the context of polynomial, clean, and nil-clean rings. We investigate whether the properties defining fine rings are preserved under polynomial extensions and how these rings align or contrast with clean and nil-clean structures. Through this study, we seek to contribute to the deeper understanding of ring decomposition theories and open avenues for further research in ring theory.

## 2. Preliminaries

**Definition (2.1) [8], [11]:**

An element a of a ring R is deemed clean if $a = e + u$, where $e \in \mathrm{Idem}(R)$ and $u \in U(R)$. When ∀ a ∈ R is clean, the ring will be clean. If $eu = ue$, we refer to the ring as strongly clean.

**Definition (2.2) [5]:**

If $e \in \mathrm{Idem}(R)$ and $t \in \mathrm{Nil}(R)$ so that $r = e + t$, then an element r in R is said to be nil-clean. If it is possible to choose an idempotent and nilpotent such that $te = et$, the element r is further referred to as strongly nil-clean. If each of the elements in a ring is nil-clean, the ring is said to be nil-clean **[9]**.

**Definition (2.3) [4]:**

An element $0 \neq a \in R$ is called fine if $a = u + t$, where $u \in U(R)$ and $t \in \mathrm{Nil}(R)$. a nonzero ring R is called a fine ring if all of the nonzero elements are fine.

**Definition (2.4) [12], [1]:**

If the maximum ideal of the ring R is unique, it is referred to as a local ring.

**Definition (2.5) [7]:**

If each element in the ring is a unit element, the ring R is referred to as a division ring.

**Definition (2.6) [3]:**

If the ring R contains solely trivial ideals, it is referred to as a simple ring.

**Notes (2.7):**

- **Note 1:** R is a simple ring if it is a fine one [4]:

  Proof: Suppose that R is a fine ring. Then $\forall\, 0 \neq a \in R$, there exist $u \in U(R)$and $t \in \mathrm{Nil}(R)$ as follows: $a = u + t$. Since u is a unit, there exists $u^{-1} \in R$ such that: $u^{-1}a = 1 + u^{-1}t$. Set $s = u^{-1}t$. Then s is nilpotent, and hence 1+s unit in R; $u^{-1}a \in U(R)$. Multiplying by u, we obtain $a = u(u^{-1}a) \in U(R)$.

Thus, every nonzero element of R is a unit, which implies that R is a division ring. Particular, R has no nontrivial ideals, and hence R is a simple ring.

- **Note 2:** Each division ring is a fine ring:

  Proof: If we take a division ring D, which is not zero, $\forall\ 0 \neq a \in D$, we will show that: $a = u + t,\ u \in U(D),\ t \in Nil(D)$. Since each element that is not zero is a unit in D, then $U(D) = D \setminus \{0\}$. Take $t \in D;\ t^n = 0$, for some $n \geq 1$. There for $t^n = 0 \rightarrow t = 0$ (the division ring does not contain zero divisors). And with this: Nil(D)={0}. Now, $\forall\ 0 \neq a \in D \rightarrow a = a + 0 = u + t;\ u = a \in U(D), t = 0 \in Nil(D)$, so D is a fine ring.

  In other words, since an element $0 \neq a \in D$ is fine iff $a \in U(D)$, if $a \in D \setminus (0)$ possesses a fine decomposition u+t, then $a = u(1 + u^{-1}t) \in U(D)$, by $(ut = tu \rightarrow u^{-1}t \in Nil(D))$. This gives D is fine.

**Definition (2.8) [2]**:

If the ring R does not contain any nonzero nilpotent elements, it is referred to as a reduced ring.

**Definition (2.9) [13]:**

Consider R to be a ring. Formally, an R- coefficient polynomial and indeterminate x is a sum:

$f(x) = \sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + \cdots + a_n x^n$, where $n \geq 0$, is an integer and each $a_k \in R$. R[x], often known as all such polynomials are contained in the polynomial ring in the indeterminate x over the ring R. [6], [10]

**Theory (2.10):** If a ring R=Q then it's fine ring:

Proof: Since R=Q and it's field then, $\forall 0 \neq a \in R\ \exists\ a = u + t$, where $u \in U(R), t \in Nil(R)$. Every element in Q is reversible, so $U(R) = Q \setminus \{0\}$.

Now, we will show that the only nilpotent element in Q is zero, let $t \in Q;\ t^n = 0, n \in N$ because Q not contain zero divisors (ab=0; a=0 or b=0) then, $t^n = t.t \ldots . t = 0 \ \rightarrow t = 0 \quad \therefore Nil(Q) = \{0\}$.

Now, $\forall\ a \in Q\ \rightarrow a = a + 0$, that's mean Q is fine ring.

**Corollary (2.11):** Every field with the form $z_p$, where p is prime, is a fine ring.

## 3. Propositions

**Proposition (3.1):** Let R be a nonzero ring. Then R[x] isn't a fine ring. Proof: Consider the ideal $(x) = \{xf(x) | f(x) \in R[x]\}$. This ideal is nonzero (since $x \neq 0$) and it's a proper (since $1 \notin (x)$). Thus, R[x] has a nontrivial ideal, so it is not a simple ring. Since every fine ring must be simple, it follows that R[x] is not fine.

In other words, since $\mathrm{Nil}(R[x]) = \{\, a_0 + a_1x + \cdots + a_nx^n | a_i \in R, (i = 1, 2, \dots, n)\}$ and $\mathrm{UR}[x] = a_0 + a_1x + \cdots + a_nx^n | a_0 \in U(R), a_i \in R\}$, if the element x is a fine, we may let: $x = u + t = t + (u_1 + a_1x + \cdots) + (u_2 + a_2x + \cdots) + (u_n + a_nx + \cdots)$, where $t \in \mathrm{Nil}(R)$, $u_i \in U(R)$ and $a_i \in R$, $\forall\ 1 \leq i \leq n$, then $\sum_{i=1}^{n} a_i = 1 \in R$. That means it is not a nilpotent element and not a unit element. So R[x] does not contain u and t, which is a contradiction. Consequently, R[x] isn't a fine ring.

**Notes:** If we assume f(x)=x ∈ Z[x], we may now take $x = u + t$ such that u is unit in Z[x] and t is nilpotent in Z[x].

We see that:

- Z[x] only has units of ±1.
- Since only {0} are nilpotent elements in Z[x] then, Z[x] is a reduced ring.
- The element x cannot be represented as the sum of a unit and nilpotent. Z[x] is therefore not a fine ring.
- The reality R is a fine does not necessarily mean that R[x] will also be a fine because polynomials introduce new elements (like x) that cannot be represented as needed.
- When R is simple, then R[x] is not simple, which goes against one of the most crucial characteristics of simple rings, which is that they are simple.
- R[x] not simple ring, because it's contain a non-trivial ideal and so that R[x] not fine.[14], [15]

**Proposition (3.2):** If R= $F_2[x]/(x^2)$, then R satisfies the following:

1. $F_2[x]/(x^2)$ is clean ring.
2. $F_2[x]/(x^2)$ is nil-clean ring

Proof:

1. Since 1×1=1 and $(1+x)^2=1+2x+x^2=1$, we have {0, 1, x, 1+x} are the elements of R; $x^2=0$, indicating that {1, 1+x} are unit elements in R and that {0, 1} are idempotent elements. Such that if $e = a + bx \rightarrow \quad e^2 = a \rightarrow a + bx = a \rightarrow b=0 \rightarrow \mathrm{Idem}(R)=\{0,1\}$, A unit is an element $a + bx \in R$ if and only if $a \neq 0$ then, U(R)={1, 1+x}.

   We shall now demonstrate that any element in R may be expressed as: unit plus idempotent:

   - 0 =1+ (1+0) =1+1 → 0=1+1.
   - 1=1+0
   - x=1+ (1+x).
   - 1+x=0+ (1+x).

Then, $R= F_2[x]/(x^2)$ is clean.

2. The ring $R= F_2[x]/(x^2)$ is nil-clean if we have $\forall\, r \in R$ then, $r=e+t$, where $e^2=e$ and t nilpotent element. Since $a+bx$ $(a, b \in F_2,\ x^2 = 0)$ then, $(a + bx)^2 = a^2$ in $F_2$; $2ab=0$, $x^2 = 0 \rightarrow (a + bx)^2 = a$. If $a=0$ then, a is a nilpotent $\rightarrow \mathrm{Nil}(R) = \{0, x\}$. From (1) we have $\mathrm{Idem}(R)=\{0, 1\}$, so that: $0=0+0$, $1=1+0$, $x=0+x$, $1+x=1+x$ $\therefore$ R is Nil-clean.

**Corollary (3.3):** The ring $R=R[x]/(x^n)$ is clean in the following cases:

- If the ring R is commutative.
- If R is local ring.
- If R is a well-structured simple ring.

**A Special Case of a Fine Ring (3.4):**

The ring $R= F_q[x]/(x^n)$ is not fine because it's not simple ring ( contains a non-trivial ideal which is (x) ).

Now, we will try to show the possibility of making it fine ring by the following steps:

1. Dividing by reduction $R/J(R)$, $J(R) = (x)$. We will get $(F_q[x]/(x^n))/(x)) \cong F_q$ and by (2.10, 2.11) it's fine ring.

   Such that: $J(R) = \cap\, \{M/M \text{ is } R'\text{s maximal ideal}\}$ or $J(R) = \{a \in R \setminus\ 1 - ra\ \in U(R) \forall\, r \in R\}$.

2. Raising to matrices: (matrix extension), $F_q[x]/(x^n)=S=M_k(F_q)$ which is fine because it's division ring. See [4]

## Summary

1. The ring $R= F_q[x]/(x^n)$ cannot be directly transformed into a fine ring, but a fine ring associated with it can be obtained either by taking the quotient by J(R) or by going to a ring of matrices over the field $F_q$. This shows that the fine property is not stable under extensions with nilpotent ideals.
2. The examples $R= F_q[x]/(x^n)$ importantly show that the clean property is broader than nil-clean, as clean is satisfied for all prime numbers, while nil-clean is only satisfied when it is $q=2$.
3.

| Ring | Fine | Clean | Nil-clean | Reason |
|---|---|---|---|---|
| Q | ✓ | ✓ | ✗ | No nonzero nilpotent elements |
| R | ✓ | ✓ | ✗ | Same reason |
| $F_q$, q is prime | ✓ | ✓ | ✗ (unless ( q=2 )) | Nil-clean iff Boolean |
| $F_2$ | ✓ | ✓ | ✓ | Boolean field |
| Q[x] | ✗ | ✗ | ✗ | Not simple, few idempotents |
| $F_q[x]/(x^n)$ | ✗ | ✓ | ✓ if ( q=2 ) | Has nilpotent elements |

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