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Free Vibration of Isotropic Material Euler-Bernoulli Beam Resting on Pasternak Foundation

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Abstract

In this paper, free vibration of isotropic material Euler-Bernoulli beam resting on Pasternak foundation is considered. The governing equations of motion for isotropic beam are derived based on energy method, and Hamilton's principle. These equations are solved using Navier's method for simply support boundary condition. The effect of spring constant (K_w), shear constant (K_G) and longitudinal wave number (m) on natural frequency (axial natural frequency and bending natural frequency) is considered. A validation is made with previous similar researches in the literature to check the accuracy of the present study. Natural frequencies are computed for various values of the parameters mentioned using MATLAB program. The beam is considered to be made of steel.

Keywords: Free Vibration, Isotropic Materials, Euler-Bernoulli Beam, Pasternack Foundation, Viscoelastic Foundation.

1. Introduction

Vibration analysis is a mechanical phenomenon when the natural frequency is near the excitation frequency, the resonance is occurred. The oscillations may be *periodic*, such as the motion of a pendulum or not periodic, like the movement of a tire on a stone way. Vibration can be classified into two types: *free* vibration and *forced* vibration. The free vibration happens where a mechanical system is beginning in movement with an initial input and permitted to oscillate freely. Forced vibration happens when an external load as a time function is applied to a mechanical system. The inconvenience can be divided into a periodic and steady type input, a transient input, or a non-periodic input. The periodic input can be a harmonic or a non-harmonic inconvenience [1].

Euler–Bernoulli beam theory (EBBT) or classical beam theory (CBT) is a facilitation of the linear theory of elasticity which enable computing the load carrying and deflection properties of beams. It involves the case for small deflections of a beam that is subjected just to side loads. It is therefore a special case of Timoshenko beam theory [2].

The Pasternak foundation is generally one of the elastic foundation models. This type of foundation can be seen as a system of closely spaced linear springs coupled to each other with elements which transfer a shear force proportional to the slope of the foundation surface. The Pasternak foundation can be seen as an overlay having a surface tension applied on a system of elastic springs as well. As a result, the connection of spring can be saved energy of structure. An intermission in slope of the displacement can occur at an edge of the beam or the plate resting on the elastic foundation [3].

A variety of researches were made on free vibration of Euler-Bernoulli and Timoshenko Beams resting on viscoelastic foundations. For example, Chen et al. [4] proposed a mixed method, which combines the state space method and the differential quadrature method for bending and free vibration of arbitrarily thick beams resting on a Pasternak elastic foundation. Based on the two-dimensional state equation of elasticity, the domain along the axial direction is discretized according to the principle of differential quadrature (DQ). Peng and Wang [5] investigated free transverse vibrations of finite Euler–Bernoulli beams resting on viscoelastic Pasternak foundations. They applied differential quadrature methods (DQ) directly to the governing equations of the free vibrations. They calculated the simple supported boundary condition, the natural frequencies of the transverse vibrations. Bezerra et al. [6] analyzed the influence of the foundation parameters and the boundary conditions configuration in the dynamic response of Euler-Bernoulli beam on Pasternak foundation. They developed a finite element solution and compared the results with the analytic solution to verify the reliability of the method in this kind of problem. Xu and Wang [7] produced a new method for analysis of transverse free vibration of a finite-length Euler-Bernoulli beam resting on a variable Pasternak elastic foundation. Schwarz distribution was adopted for simplifying the partial derivative of Dirac function and compound trapezoidal integral formula was adopted for discretizing the shape function of beam. Abbas et al. [8] studied the vibration analysis of uniform and nonuniform Euler–Bernoulli, and Timoshenko beams utilizing the differential quadrature method (DQM). The focus of this work was on studying the deflection difference between both beam theories at different beam dimensions as well as showing the shape of rotation of the cross section while applying a nodal point load equation to simulate the moving load. Free vibrations and dynamic response of a composite beam on the elastic foundation reinforced with graphene plates were studied in [9]. A beam under the action of a moving load was considered. A Newmark method was used to solve the problem of forced vibrations. The research [10] also studies free vibrations of beams placed on the elastic foundation. For this purpose, eigenfrequencies and modes are determined using analytical and numerical methods. In [11] Hermite finite element method to get the numerical approximation scheme for the vibration equation of viscoelastic Pasternak foundation beam was used. The study of Yaniutin et. al. [12] aimed to develop a mathematical vibration model for an Euler–Bernoulli isotropic elastic beam with infinite length on the two-parameter Pasternak foundation. They studied stress-strain state under the action of moving uniform loads. They solved initial differential equation using the direct and inverse Fourier transforms.

In this study, free vibration of isotropic material Euler-Bernoulli beam resting on Pasternak elastic foundation is investigated. The governing equations of motions are derived depending on energy method and Hamilton's principle.

The main contributions of this work:

1. Calculating natural frequencies and mode shape for the proposed beam.
2. Deriving the governing equation of motion for the proposed beam according to Euler-Bernoulli beam theory based on Hamilton's principle.
3. The resulting governing equations were solved by a numerical method (Navier's method) for simply supported boundary conditions. Navier's method gives accurate and exact solution
4. Studying the effect of the Pasternak foundation on the proposed beam according to natural frequency and mode shape.
5. Investigating the effects of changing spring constant (K_w), shear constant (K_G), longitudinal wave number (m), and slenderness ratio (L/h) on natural frequencies (axial natural frequency and bending natural frequency).

The rest of the paper is organized as follows. The description of the used beam and the derivation of governing equations is made in section 2. Navier's method to solve the developed equations is detailed in section 3. The numerical results for the experiments made are shown in section 4. Finally, the main conclusions and insights for future research are summarized in section 5.

2. The Governing Equations of Motion for Isotropic Euler-Bernoulli Beam

The Euler-Bernoulli Beam that considered in this study is indicated in Fig. (1). A simple supported beam manufactured of steel material with length (L), Young modulus of elasticity

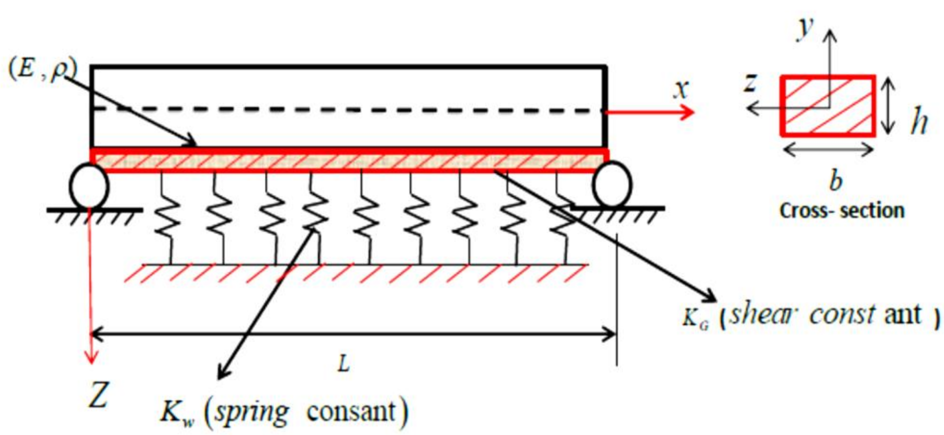

Fig (1): Free vibration for Euler-Bernoulli simply supported beam resting on Pasternak Foundation

(E), density (ρ), width (b), thickness (h), resting on Pasternak foundation with spring constant (K_w) and shear constant (K_G), as shown in Fig. (1).

The governing equations of motion for the isotropic material beam depending on "Euler-Bernoulli" beam are derived by using "Hamilton's principle" [13]. Displacement fields of Euler-Bernoulli Beam are considered as follows:

Normal strains are written as follows:

$$\left\{\underbrace{\begin{array}{l} \varepsilon_{xx} = \dfrac{\partial U(x,z,t)}{\partial x} \Rightarrow \varepsilon_{xx} = \dfrac{\partial u_o(x)}{\partial x} - z\dfrac{\partial^2 w(x)}{\partial x^2}. \\ \varepsilon_{yy} = \dfrac{\partial U}{\partial y} = 0. \\ \varepsilon_{zz} = \dfrac{\partial w}{\partial x} = 0. \end{array}}_{normal\ ,strain}\right\} \quad (2\text{-a})$$

Shear strains are written as follows:

$$\begin{array}{l} \gamma_{xy} = \dfrac{\partial U}{\partial y} + \dfrac{\partial V}{\partial x} = 0. \\ \gamma_{xz} = \dfrac{\partial U}{\partial z} + \dfrac{\partial W}{\partial x} = 0. \\ \gamma_{yz} = \dfrac{\partial V}{\partial z} + \dfrac{\partial W}{\partial y} = 0. \end{array} \quad (2\text{-b})$$

Finally, the components of strains are indicated as follows:

$$\left.\begin{array}{l} \varepsilon_{xx} = \dfrac{\partial U(x,z,t)}{\partial x} \Rightarrow \dfrac{\partial u_0(x)}{\partial x} - z\dfrac{\partial^2 w(x)}{\partial x^2}. \\ \varepsilon_{yy} = \dfrac{\partial V}{\partial y} = 0. \\ \varepsilon_{zz} = \dfrac{\partial W}{\partial z} = 0. \\ \gamma_{xy} = \dfrac{\partial U}{\partial y} + \dfrac{\partial V}{\partial x} = 0. \\ \gamma_{xz} = \dfrac{\partial U}{\partial z} + \dfrac{\partial W}{\partial x} = 0. \\ \gamma_{yz} = \dfrac{\partial V}{\partial z} + \dfrac{\partial W}{\partial y} = 0. \end{array}\right\} \quad (2\text{-c})$$

The strain energy is defined as:

$$U = \frac{1}{2}\int \sigma_{ij}\,\varepsilon_{ij}\,dv \Rightarrow \delta U = \frac{1}{2}\int [\delta\sigma_{ij}\,\varepsilon_{ij} + \sigma_{ij}\,\delta\varepsilon_{ij}]dv\,. \quad (3)$$

By taking the variation for the potential energy, we obtain:

$$\delta U = \int (\sigma_{ij}\,\delta\varepsilon_{ij})dv\,. \quad \text{(4-a)}$$

$$\delta U = \int [\sigma_{xx}\,\delta\varepsilon_{xx} + \cancel{\underbrace{\sigma_{yy}\,\delta\varepsilon_{yy}}_{=0}} + \cancel{\underbrace{\sigma_{zz}\,\delta\varepsilon_{zz}}_{=0}} + \sigma_{xz}\,\delta\varepsilon_{xz} +$$

$$\underbrace{\sigma_{yx}}_{=\sigma_{xy}}\,\delta\,\underbrace{\varepsilon_{yx}}_{=\varepsilon_{xy}} + \underbrace{\sigma_{zy}}_{=\sigma_{yz}}\,\delta\,\underbrace{\varepsilon_{zy}}_{=\varepsilon_{yz}} + \underbrace{\sigma_{zx}}_{=\sigma_{xz}}\,\delta\,\underbrace{\varepsilon_{zx}}_{=\varepsilon_{xz}}]dv\,. \quad \text{(4-b)}$$

$$\delta U = \int [\sigma_{xx}\,\delta\varepsilon_{xx} + 2\sigma_{xy}\,\delta\,\underbrace{\varepsilon_{xy}}_{\frac{\gamma_{xy}}{2}} + 2\sigma_{yz}\,\delta\,\underbrace{\varepsilon_{yz}}_{\frac{\gamma_{yz}}{2}} + 2\sigma_{xz}\,\delta\,\underbrace{\varepsilon_{xz}}_{\frac{\gamma_{xz}}{2}}]\mathrm{dv}. \quad \text{(4-c)}$$

$$\delta U = \int [\sigma_{xx}\,\delta\varepsilon_{xx}]\,\mathrm{dv}. \quad \text{(4-d)}$$

$$\delta U = \int [\sigma_{xx}\,\delta\varepsilon_{xx}]\,\mathrm{dv}. \quad \text{(4-e)}$$

Substituting Eq. (2-c) into Eq. (4-e), we obtain:

$$\delta U = \int \sigma_{xx}\left[\delta\left(\frac{\partial u_o}{\partial x}\right) - z\,\delta\left(\frac{\partial^2 w}{\partial x^2}\right)\right]dA\,dx \quad \text{(5-a)}$$

The resultant force and moment are considered as follows:

$$\therefore \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} zdz\ ,and\ , \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz \quad \text{(5-b)}$$

Therefore,

$$\left.\begin{aligned} N_x &= \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_x\,dA\,. \\ \mathrm{M}_x &= \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_x\,z\,dA\,. \end{aligned}\right\} \quad \text{(5-c)}$$

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الإصدار (5)، العدد (8)

$$N_x \frac{\partial}{\partial x}(\delta u_0) = \frac{\partial}{\partial x}(N_x \delta u_0) - \frac{\partial N_x}{\partial x}\delta u_0.$$
$$M_x \frac{\partial}{\partial x}(\delta \frac{\partial w}{\partial x}) = \frac{\partial}{\partial x}(M_x \delta(\frac{\partial w}{\partial x})) - \frac{\partial M_x}{\partial x}\delta(\frac{\partial w}{\partial x}). \quad (6)$$
$$\frac{\partial M_x}{\partial x}\frac{\partial}{\partial x}(\delta \dot{w}) = \frac{\partial}{\partial x}(\frac{\partial M_x}{\partial x}\delta(w)) - \frac{\partial^2 M_x}{\partial x^2}\delta(w).$$

Substituting Eq. (5-c) into Eq. (4-e), we obtain:

$$\delta \mathrm{U} = \int \left[N_x \delta\left(\frac{\partial u_0}{\partial x}\right) - M_x \delta\left(\frac{\partial^2 w}{\partial x^2}\right)\right] dx . \quad (7)$$

Substituting Eq. (6) into Eq. (7), we obtain:

$$\delta U = -\int \left[\frac{\partial N_x}{\partial x}\delta u_0 + \frac{\partial^2 M_x}{\partial x^2}\delta w \right] dx \quad (8)$$

The kinetic Energy of EBBT is considered as the following form:

$$T = \int \frac{1}{2}\rho \vec{v}\,\vec{v} dv \quad (9\text{-a})$$

$$T = \frac{1}{2}\int \rho \left(\frac{\partial U(x,z,t)}{\partial t}\vec{i} + \frac{\partial V(x,z,t)}{\partial t}\vec{j} + \frac{\partial W(x,z,t)}{\partial t}\vec{k} \right). \left(\frac{\partial U(x,z,t)}{\partial t}\vec{i} + \frac{\partial V(x,z,t)}{\partial t}\vec{j} + \frac{\partial W(x,z,t)}{\partial t}\vec{k} \right) dv . \quad (9\text{-b})$$

$$\delta T = \frac{1}{2}\int \rho \left[\left(\frac{\partial U}{\partial t}\vec{i} + \frac{\partial V}{\partial t}\vec{j} + \frac{\partial W}{\partial t}\vec{k} \right).\delta\left(\frac{\partial U}{\partial t}\vec{i} + \frac{\partial V}{\partial t}\vec{j} + \frac{\partial W}{\partial t}\vec{k} \right) + \delta\left(\frac{\partial U}{\partial t}\vec{i} + \frac{\partial V}{\partial t}\vec{j} + \frac{\partial W}{\partial t}\vec{k} \right).\left(\frac{\partial U}{\partial t}\vec{i} + \frac{\partial V}{\partial t}\vec{j} + \frac{\partial W}{\partial t}\vec{k} \right) dv \quad (9\text{-c})$$

$$\delta T = \int \rho \left(\frac{\partial U}{\partial t}\vec{i} + \frac{\partial V}{\partial t}\vec{j} + \frac{\partial W}{\partial t}\vec{k} \right).\delta\left(\frac{\partial U}{\partial t}\vec{i} + \frac{\partial V}{\partial t}\vec{j} + \frac{\partial W}{\partial t}\vec{k} \right) dv \quad (9\text{-d})$$

$$\delta T = \int \rho \left[\frac{\partial U}{\partial t}\delta\left(\frac{\partial U}{\partial t}\right) + \frac{\partial V}{\partial t}\delta\left(\frac{\partial V}{\partial t}\right) + \frac{\partial W}{\partial t}\delta\left(\frac{\partial W}{\partial t}\right) \right] dv \quad (9\text{-e})$$

Substituting Eq. (1) in Eq. (9-e), we obtain:

$$\delta T = \int \left[\rho \frac{\partial u_0}{\partial t}\delta\left(\frac{\partial u_0}{\partial t}\right) - \rho z \frac{\partial u_0}{\partial t}\delta\left(\frac{\partial^2 w}{\partial t \partial x}\right) - \rho z \frac{\partial^2 w}{\partial t \partial x}\delta\left(\frac{\partial u_0}{\partial t}\right) + \rho z^2 \frac{\partial^2 w}{\partial x \partial t}\delta\left(\frac{\partial^2 w}{\partial x \partial t}\right) + \rho \frac{\partial w}{\partial t}\delta\left(\frac{\partial w}{\partial t}\right) \right] dz dA \quad (10)$$

The mass moment inertia is defined as:

$$where: \mathrm{I}^{(i)} = \int \rho z^{(i)} dz, i = 0,1,2. \tag{11}$$

Substituting Eq. (11) into Eq. (10), we obtain:

$$\delta T = \int \Bigg[\mathrm{I}^{(0)} \frac{\partial u_0}{\partial t} .\delta\left(\frac{\partial u_0}{\partial t}\right) - \mathrm{I}^{(1)} \frac{\partial u_0}{\partial t} \delta\left(\frac{\partial^2 w}{\partial t\, \partial x}\right) - \mathrm{I}^{(1)} \frac{\partial^2 w}{\partial t\, \partial x} \delta\left(\frac{\partial u_0}{\partial t}\right) + \mathrm{I}^{(2)} \frac{\partial^2 w}{\partial t\, \partial x} \delta\left(\frac{\partial^2 w}{\partial t\, \partial x}\right) + \mathrm{I}^{(0)} \frac{\partial w}{\partial t} \delta\left(\frac{\partial w}{\partial t}\right) \Bigg] dA \tag{12}$$

Therefore:

$$\left.\begin{aligned}
&\frac{\partial u_0}{\partial t} \delta\left(\frac{\partial u_0}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial u_0}{\partial t} \delta u_0\right) - \frac{\partial^2 u_0}{\partial t^2} \delta u_0. \\
&\frac{\partial u_0}{\partial t} \delta\left(\frac{\partial^2 w}{\partial t\, \partial x}\right) = \frac{\partial}{\partial t}\left(\frac{\partial u_0}{\partial t} \delta\left(\frac{\partial w}{\partial x}\right)\right) - \frac{\partial^2 u_0}{\partial t^2} \delta\left(\frac{\partial w}{\partial x}\right). \\
&\frac{\partial w}{\partial t} \delta\left(\frac{\partial w}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t} \delta(w)\right) - \frac{\partial^2 w}{\partial t^2} \delta(w).
\end{aligned}\right\} \tag{13}$$

Substituting Eq. (13) into Eq. (12), we obtain:

$$\delta T = \int \Bigg[-\mathrm{I}^{(0)} \frac{\partial^2 u_0}{\partial t^2} \delta u_0 + \mathrm{I}^{(1)} \left(\frac{\partial}{\partial x} \left(\underbrace{\frac{\partial^2 u_0}{\partial t^2} \delta(w)}_{=0} \right) - \frac{\partial^3 u_0}{\partial x\, \partial t^2} \delta(w) \right) + \mathrm{I}^{(1)} \frac{\partial^3 w}{\partial t^2 \partial x} \delta u_0 - \mathrm{I}^{(2)} \left(\frac{\partial}{\partial x} \left(\underbrace{\frac{\partial^3 w}{\partial x\, \partial t^2} \delta w}_{=0} \right) - \frac{\partial^4 w}{\partial x^2 \partial t^2} \delta w \right) - \mathrm{I}^{(0)} \frac{\partial^2 w}{\partial t^2} \delta w \Bigg] dA. \tag{14-a}$$

Where dA=bdx, b is the width of the beam. Therefore, the final kinetic energy formula is obtained as follows:

$$\delta T = \int \Bigg[-\mathrm{I}^{(0)} \frac{\partial^2 u_0}{\partial t^2} \delta u_0 - \mathrm{I}^{(1)} \frac{\partial^3 u_0}{\partial x\, \partial t^2} \delta w + \mathrm{I}^{(1)} \frac{\partial^3 w}{\partial x\, \partial t^2} \delta u_0 + \mathrm{I}^{(2)} \frac{\partial^4 w}{\partial x^2 \partial t^2} \delta w - \mathrm{I}^{(0)} \frac{\partial^2 w}{\partial t^2} \delta w \Bigg] bdx. \tag{14-b}$$

The external work is defined as:

$$W_{ext} = -\frac{1}{2} \int F_{ext}(x) w(x) dA \tag{15}$$

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By taking the variation for external work, we obtain:

$$\delta W_{ext} = -\frac{1}{2}\int [F(x)\delta w(x) + \delta F(x)\, w(x)]dA \tag{16-a}$$

$$\delta W_{ext} = -\frac{1}{2}\int [F_{ext}(x)\; \delta w(x,t)]bdx \tag{16-b}$$

$$F_{ext}(x) = \underbrace{q(\,x,t)}_{\text{Forced Vibration}} \quad + \quad \underbrace{F_P}_{\text{Pasternak foundation}} \tag{17}$$

Substituting Eq. (17) into Eq. (16-b), we obtain:

$$\delta W_{ext} = -\int (q(x,t) + F_p)\delta w\,(\mathrm{x,t})bdx \tag{18}$$

In this thesis, free vibration analysis of EBBT is considered, then q(x, t) is equal to zero.

By applying Hamilton's principle, we have:

$$\prod = T - (U + W_{ext}) \tag{19}$$

Where:

$\prod$: Total potential energy.

T: kinetic energy.

U: potential energy.

W_{ext}: external work.

By taking variation for total potential energy and making it equal to zero, we obtain:

$$\delta\prod = 0 \Rightarrow \delta T - (\delta U + \delta W_{ext}) = 0 \tag{20}$$

By integrating Eq. (20), we can obtain as follows:

$$\int [\delta T - (\delta U + \delta W_{ext})]\mathrm{dt} = 0 \tag{21}$$

By substituting Eqs. (8), (14-b), (18) into Eq. (21), we obtain:

$$\begin{aligned}\int [\Big(& -\mathrm{I}^{(0)}\frac{\partial^2 u_0}{\partial t^2}\delta u_0 - \mathrm{I}^{(1)}\frac{\partial^3 u_0}{\partial x\,\partial t^2}\delta w + \mathrm{I}^{(1)}\frac{\partial^3 w}{\partial x\,\partial t^2}\delta u_0 + \mathrm{I}^{(2)}\frac{\partial^4 w}{\partial x^2 \partial t^2}\delta w - \\ & \mathrm{I}^{(0)}\frac{\partial^2 w}{\partial t^2}\delta w \Big)b + \frac{\partial N_x}{\partial x}\delta u_0 + \frac{\partial^2 M_x}{\partial x^2}\delta w + q(x,t)\delta w b + F_p \delta w b\,]dx = 0\end{aligned} \tag{22-a}$$

$$\left.\begin{aligned}&\delta u_0 : -I^{(0)}\frac{\partial^2 u_0}{\partial t^2}b + I^{(1)}\frac{\partial^3 w}{\partial x\,\partial t^2}b + \frac{\partial N_x}{\partial x} = 0\\ &\delta w \ : -\mathrm{I}^{(1)}\frac{\partial^3 u_0}{\partial x\,\partial t^2}b + I^{(2)}\frac{\partial^4 w}{\partial x^2\partial t^2}b - I^{(0)}\frac{\partial^2 w}{\partial t^2}b + \frac{\partial^2 M_x}{\partial x^2} + q(x,t)b + F_p b = 0\end{aligned}\right\} \quad (22\text{-b})$$

From Eq. (5-c), we have:

$$N_x = \int \sigma_x dA = \int E\,\varepsilon_x dA \rightarrow \int_{\frac{-h}{2}}^{\frac{h}{2}} E\left(\frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\right)bdz = Eb\frac{\partial u_0}{\partial x}z\bigg|_{\frac{-h}{2}}^{\frac{h}{2}}$$

$$-Eb\frac{\partial^2 w}{\partial x^2}\frac{z^2}{2}\bigg|_{\frac{-h}{2}}^{\frac{h}{2}} \Rightarrow N_x = Ebh\frac{\partial u_0}{\partial x} \quad (23\text{-a})$$

$$\frac{\partial N_x}{\partial x} = Ebh\frac{\partial^2 u_0}{\partial x^2}\ldots\ldots\ldots\ldots\ldots\ldots(2.22) \quad (23\text{-b})$$

$$\mathrm{M}_x = \int \sigma_x z dA = \int E\,\varepsilon_x z\ \mathrm{dA} \xrightarrow[2]{sub} \int_{\frac{-h}{2}}^{\frac{-h}{2}} Eb\left(\frac{\partial u_0}{\partial x}z - z^2\frac{\partial^2 w}{\partial x^2}\right)dz$$

$$= Eb\frac{\partial u_0}{\partial x}\frac{z^2}{2}\bigg|_{\frac{-h}{2}}^{\frac{h}{2}} - Eb\frac{\partial^2 w}{\partial x^2}\frac{z^3}{3}\bigg|_{\frac{-h}{2}}^{\frac{h}{2}} \Rightarrow M_x = \frac{-Ebh^3}{12}\frac{\partial^2 w}{\partial x^2} \quad (24\text{-a})$$

$$\frac{\partial^2 M}{\partial x^2} = \frac{-Ebh^3}{12}\frac{\partial^4 w}{\partial x^4} \quad (24\text{-b})$$

By substituting Eqs. (24-b) and (23-b) into Eq. (22-b), we obtain:

$$\delta u_0 : -\mathrm{I}^{(0)}b\frac{\partial^2 u_o}{\partial x^2} + \mathrm{I}^{(1)}b\frac{\partial^3 w}{\partial x\,\partial t^2} + Ebh\frac{\partial^2 u_o}{\partial x^2} = 0 \quad (25)$$

$$\delta w \ : -\mathrm{I}^{(1)}b\frac{\partial^3 u_o}{\partial x\,\partial t^2} + \mathrm{I}^{(2)}b\frac{\partial^4 w}{\partial x^2\partial t^2} - \mathrm{I}^{(0)}b\frac{\partial^2 w}{\partial t^2} - \frac{Eh^3}{12}b\frac{\partial^4 w}{\partial x^4} + qb + F_p b = 0 \quad (26)$$

For free vibration: q=0

$$\therefore F_p = -k_w w + k_G \nabla^2 w \quad (27\text{-a})$$

K_W: spring constant.

K_G: shear constant of Pasternak.

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \cancel{\frac{\partial^2}{\partial y^2}} + \cancel{\frac{\partial^2}{\partial z^2}} \quad (27\text{-b})$$

$$\therefore F_p = -k_w W + k_G\frac{\partial^2 w}{\partial x^2} \quad (27\text{-c})$$

By dividing the both sides of Eqs. (25) and (26) on width of beam (b) and addition of Eq. (27-c) to

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(26), we obtain the following equations of motion for isotropic EBBT.

$$\delta u_0 : -\mathrm{I}^{(0)} \frac{\partial^2 u_o}{\partial x^2} + \mathrm{I}^{(1)} \frac{\partial^3 w}{\partial x \, \partial t^2} + Eh \frac{\partial^2 u_o}{\partial x^2} = 0 \tag{28}$$

$$\delta w \; : -\mathrm{I}^{(1)} \frac{\partial^3 u_o}{\partial x \, \partial t^2} + \mathrm{I}^{(2)} \frac{\partial^4 w}{\partial x^2 \partial t^2} - \mathrm{I}^{(0)} \frac{\partial^2 w}{\partial t^2} - \frac{Eh^3}{12} \frac{\partial^4 w}{\partial t^4} - k_w w + k_G \frac{\partial^2 w}{\partial x^2} = 0......(\tag{29}$$

We calculate the parameters I $^{(0)}$, I $^{(1)}$, I $^{(2)}$ by limited integration, we obtain:

$$\left.\begin{aligned} \mathbf{I}^{(0)} &= \int_{\frac{-h}{2}}^{\frac{h}{2}} \rho dz = \rho h \\ \mathbf{I}^{(1)} &= \int_{\frac{-h}{2}}^{\frac{h}{2}} \rho z dz = 0 \\ \mathbf{I}^{(2)} &= \int_{\frac{-h}{2}}^{\frac{h}{2}} \rho z^2 dz = \frac{1}{12} \rho h^3 \end{aligned}\right\} \tag{30}$$

By substituting Eq. (30) into Eq. (28) and Eq. (29), we obtain:

$$\delta u_0 : -\rho h \frac{\partial^2 u_0}{\partial t^2} + (0) \frac{\partial^3 w}{\partial x \, \partial t^2} + Eh \frac{\partial^2 u_0}{\partial x^2} = 0 \tag{31-a}$$

$$-\rho h \frac{\partial^2 u_0}{\partial t^2} + Eh \frac{\partial^2 u_0}{\partial x^2} = 0 \tag{31-b}$$

$$\delta w \; : -(0) \frac{\partial^3 u_0}{\partial x \, \partial t^2} + \frac{1}{12} \rho h^3 \frac{\partial^4 w}{\partial x^2 \partial t^2} - \rho h \frac{\partial^2 w}{\partial t^2} - \frac{Eh^3}{12} \frac{\partial^4 w}{\partial x^4} - k_w w + k_G \frac{\partial^2 w}{\partial x^2} = 0 \tag{31-c}$$

$$\therefore \frac{1}{12} \rho h^3 \frac{\partial^4 w}{\partial x^2 \partial t^2} - \rho h \frac{\partial^2 w}{\partial t^2} - \frac{Eh^3}{12} \frac{\partial^4 w}{\partial x^4} - k_w w + k_G \frac{\partial^2 w}{\partial x^2} = 0 \tag{31-d}$$

We rearrange the equations ($\delta u_0 , \delta w$) Eqs. (31-b) and (31-d), we obtain:

$$\delta u_0 : Eh \frac{\partial^2 u_0}{\partial x^2} = \rho h \frac{\partial^2 u_0}{\partial t^2} \tag{32}$$

$$\delta w \; : \frac{Eh^3}{12} \frac{\partial^4 w}{\partial x^4} + k_w w - k_G \frac{\partial^2 w}{\partial x^2} = -\rho h \frac{\partial^2 w}{\partial t^2} + \frac{1}{2} \rho h^3 \frac{\partial^4 w}{\partial x^2 \partial t^2} \ldots \tag{33}$$

The Equations (32) and (33) represent the governing equations of motion for free vibration isotropic material beam resting on Pasternak foundation.

3. Numerical Method Solution

The governing equations of motion (32) and (33) are solved semi-analytical solution using Navier's Method for the beam with simply supported boundary conditions.

In Fig. (2), an isotropic Euler-Bernoulli simply supported beam with its boundary conditions is indicated. As it is seen in Fig. (2), the boundary conditions are:

When x=0: w=0, $\frac{\partial u}{\partial x} = 0$

And when x=L: w=0, $\frac{\partial u}{\partial x} = 0$

The Navier's method for isotropic EBBT is defined as follows:

$$u(x,t) = \sum_{m=1,2}^{\infty} U_m \cos\left(\frac{m\pi x}{L}\right) e^{i\omega t} \tag{34}$$

$$w(x,t) = \sum_{m=1,2}^{\infty} W_m \sin\left(\frac{m\pi x}{L}\right) e^{i\omega t} \tag{35}$$

Where m is the axial wave numbers.

By substituting Eq. (34) into Eq. (32), we obtain:

$$\sum \underbrace{\left[\mathrm{I}^{(0)} W^{2} U_m - \mathrm{I}^{(1)} W^{2} \left(\frac{m\pi}{L}\right) W_m - Eh \left(\frac{m\pi}{L}\right)^2 U_m \right]}_{=0} \cos\left(\frac{m\pi x}{L}\right) e^{iwt} = 0 \tag{36-a}$$

$$\mathrm{I}^{(0)} W_m{}^{2} U_m - \mathrm{I}^{(1)} W^{2} \left(\frac{m\pi}{L}\right) W_m - Eh \left(\frac{m\pi}{L}\right) U_m = 0 \tag{36-b}$$

Similarly, by substituting Eq. (35) into Eq. (33), we get:

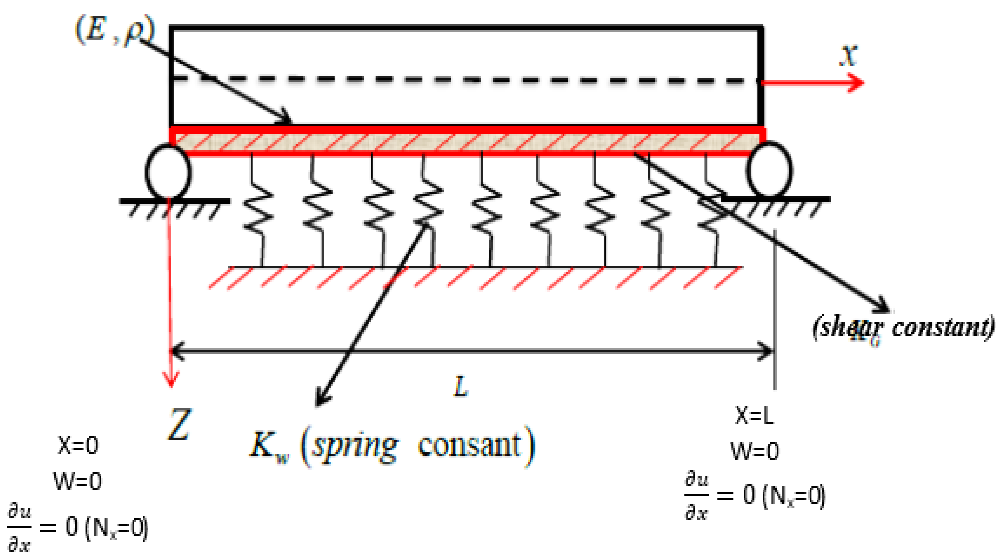

Fig (2): The schematic isotropic Euler-Bernoulli simply supported boundary condition beam resting on Pasternak Foundation.

$$\sum \begin{array}{l}[-I^{(1)}w^{2}\left(\frac{m\pi}{L}\right)U_m + I^{(2)}w^{2}\left(\frac{m\pi}{L}\right)^2 W_m + I^{(0)}w^{2}W_m \\ -\frac{Eh^3}{12}\left(\frac{m\pi}{L}\right)^2 W_m - K_w W_m - K_G\left(\frac{m\pi}{L}\right)^2 W_m]\underbrace{\sin\left(\frac{m\pi x}{L}\right)e^{iwt}}_{\neq 0} = 0\end{array} \tag{37-a}$$

$$\begin{array}{l}-\mathrm{I}^{(1)}W^{2}\left(\frac{m\pi}{L}\right)U_m + \mathrm{I}^{(2)}W^{2}\left(\frac{m\pi}{L}\right)^2 W_m + \mathrm{I}^{(0)}W^{2}W_m - \frac{Eh^3}{12}\left(\frac{m\pi}{L}\right)^2 W_m \\ -K_w W_m - K_G\left(\frac{m\pi}{L}\right)^2 W_m = 0.........(2.35)\end{array} \tag{37-b}$$

From Based on equation (36-b) and equation (37-b), we formulate a matrix as follows: (38)

$$\mathrm{I}^{(0)}W_n{}^{2}U_m - \mathrm{I}^{(1)}W^{2}\left(\frac{m\pi}{L}\right)W_m - Eh\left(\frac{m\pi}{L}\right)U_m = 0$$

$$\begin{array}{l}-\mathrm{I}^{(1)}W^{2}\left(\frac{m\pi}{L}\right)U_m + \mathrm{I}^{(2)}W^{2}\left(\frac{m\pi}{L}\right)^2 W_m + \mathrm{I}^{(0)}W^{2}U_m - \frac{Eh^3}{12}\left(\frac{m\pi}{L}\right)^2 W_m \\ -K_w W_m - K_G\left(\frac{m\pi}{L}\right)^2 W_m = 0\end{array} \tag{39}$$

$$\begin{Bmatrix} \underbrace{\mathrm{I}^{(0)}W^{2} - Eh\left(\frac{m\pi}{L}\right)^2}_{\alpha} & \underbrace{\cancel{-\mathrm{I}^{(1)}W^{2}}\left(\frac{m\pi}{L}\right)}_{=0} \\ \underbrace{\cancel{-\mathrm{I}^{(1)}W^{2}}\left(\frac{m\pi}{L}\right)}_{=0} & \underbrace{\mathrm{I}^{(2)}W^{2}\left(\frac{m\pi}{L}\right)^2 + \mathrm{I}^{(0)}W^{2} - \frac{Eh^3}{12}\left(\frac{m\pi}{L}\right)^4 - K_w - K_G\left(\frac{m\pi}{L}\right)^2}_{\beta} \end{Bmatrix} \begin{Bmatrix} U_m \\ W_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{40}$$

Substituting Eq. (30) into Eq. (40), we obtain a new form of matrix as follows:

$$\begin{Bmatrix} \alpha & 0 \\ 0 & \beta \end{Bmatrix} \begin{Bmatrix} U_m \\ W_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{41}$$

By taking the determinant for the matrix in Eq. (41), we obtain

$$\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} = 0 \Rightarrow (\alpha\beta) - (0 \times 0) = 0 \Rightarrow \therefore \alpha\beta = 0.$$

$$\therefore \alpha\beta = 0 \Rightarrow$$
$$Either:$$
$$\alpha = 0$$
$$\therefore \mathrm{I}^{(0)}\omega^{(2)} - Eh\left(\frac{m\pi}{L}\right)^2 = 0, \mathrm{I}^{(0)} = \rho h \tag{42-a}$$

$$\rho h\omega^2 - Eh\left(\frac{m\pi}{L}\right)^2 = 0 \Rightarrow \rho h\omega^2 = Eh\left(\frac{m\pi}{L}\right)^2 \tag{42-b}$$

$$\omega^2 = \omega_a^2 = \frac{Eh\left(\frac{m\pi}{L}\right)^2}{\rho h} \tag{43}$$

Where Eq. (43) represent axial natural frequency for free vibration isotropic Euler-Bernoulli beam resting on Pasternak foundation.

$$Or:$$
$$\beta = 0$$
$$\therefore I^{(2)}\omega^2\left(\frac{m\pi}{L}\right)^2 + I^{(0)}\omega^2 - \frac{Eh^3}{12}\left(\frac{m\pi}{L}\right)^4 - K_w - K_G\left(\frac{m\pi}{L}\right)^2 = 0 \tag{44-a}$$

$$\therefore I^{(2)}\omega^2\left(\frac{m\pi}{L}\right)^2 + I^{(0)}\omega^2 = \frac{Eh^3}{12}\left(\frac{m\pi}{L}\right)^4 + K_w + K_G\left(\frac{m\pi}{L}\right)^2 \tag{44-b}$$

$$\omega^2\left[I^{(2)}\left(\frac{m\pi}{L}\right)^2 + I^{(0)}\right] = \frac{Eh^3}{12}\left(\frac{m\pi}{L}\right)^4 + K_w + K_G\left(\frac{m\pi}{L}\right)^2 \tag{44-c}$$

Where:

$$I^{(0)} = \rho h, I^{(2)} = \frac{1}{12}\rho h^3$$

$$\therefore \omega^2 = \omega_b^2 = \frac{\frac{Eh^3}{12}\left(\frac{m\pi}{L}\right)^4 + K_w + K_G\left(\frac{m\pi}{L}\right)^2}{\frac{\rho h^3}{12}\left(\frac{m\pi}{L}\right)^2 + \rho h} \tag{45}$$

Where Eq. (45) represents bending natural frequency for free vibration isotropic Euler-Bernoulli beam resting on Pasternak foundation.

4. Numerical Results and Discussion

The examples are taken to observe the effects of changing spring constant (k_w), shear constant (k_G) and longitudinal wave number (m) on natural frequency (axial natural frequency ω_a and bending natural frequency (ω_b). Where natural frequencies were computed for different values of the parameters mentioned using MATLAB program. Also, the mode shapes are computed for different values of longitudinal wave number (m).

The material of the beam is steel with the properties: (Young modulus of elasticity E=200 GPa and density ρ=2860 kg/m^3). In Euler-Bernoulli beam theory, the length of the beam (L) must be greater than the height of the beam (h), this is very important in design and manufacturing of the Euler-Bernoulli beam (L>h).

Fig. (3) Indicates the axial and bending natural frequencies for different values of the ratio *L/h*. The parameters considered in this figure are: the length of the beam is proposed to be *L*=1 m, the height h=0.1, thickness (width) b=0.1m and longitudinal wave number m=1. As it is noticed in the figure, the bending natural frequency takes the shape of straight line and the axial natural frequency take the shape of cosine wave. This due to the fact that the natural oscillation of the beam is occurred without any external force or load.

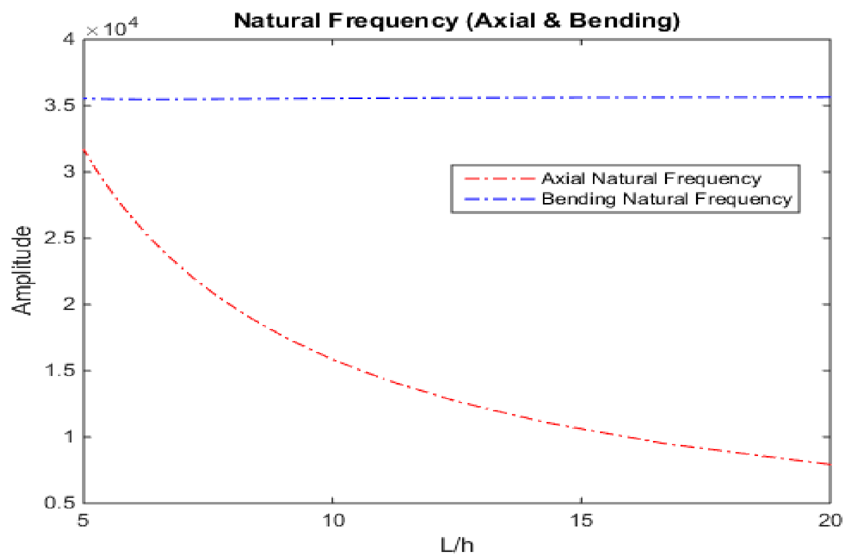

Fig. (3) Axial and bending natural frequencies versus ratio *L/h*.

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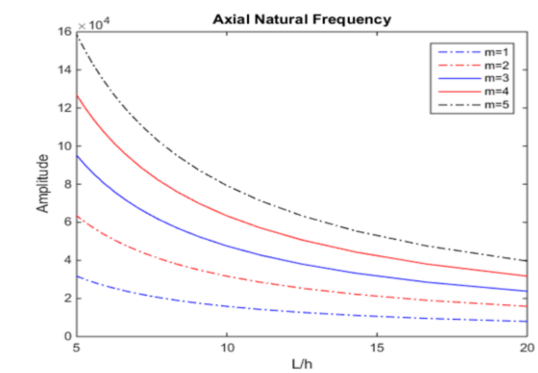
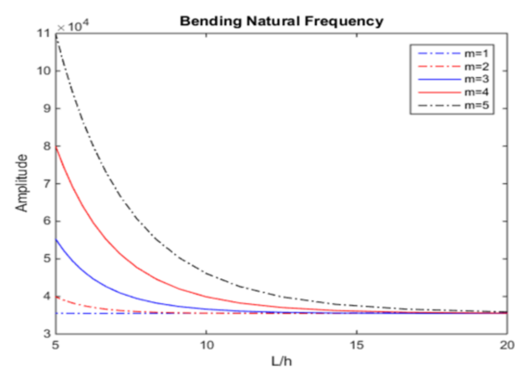
Vol. (5), No. (8) August 2026 الإصدار (5)، العدد (8)



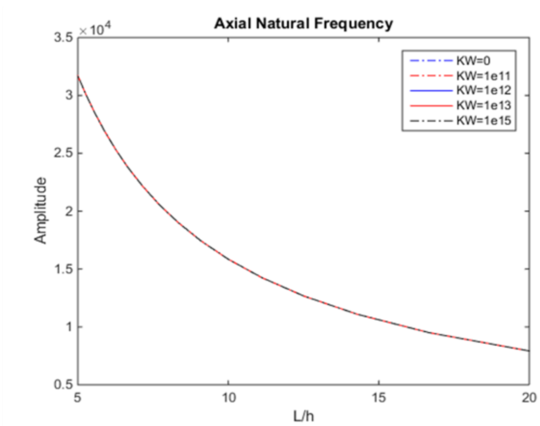
(a)



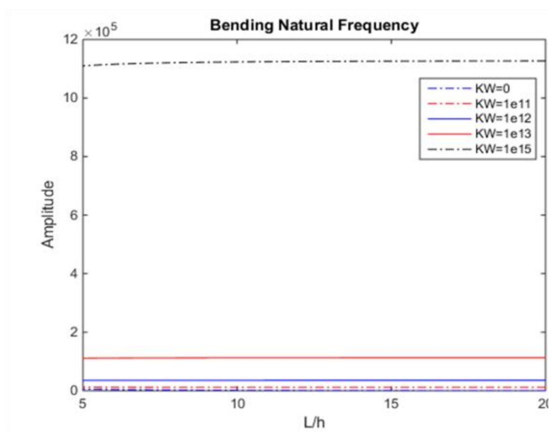
(b)

Fig. (4) Natural frequency versus ratio L/h for different m values. a) Axial natural frequency and b) Bending natural frequency

Fig. (4) Shows the effect of various the value of longitudinal wave number (m) on natural frequencies a) axial natural frequency and b) bending natural frequency. It's noticed that the natural frequencies (axial and bending) take the shape of approximately curve wave lines, and the natural frequencies increases as the value of *m* increases. Because in an elastic medium(beam) with rigidity, gives a harmonic wave oscillation and as the value of (m) increases that leads to increasing natural frequency.



(a)



(b)

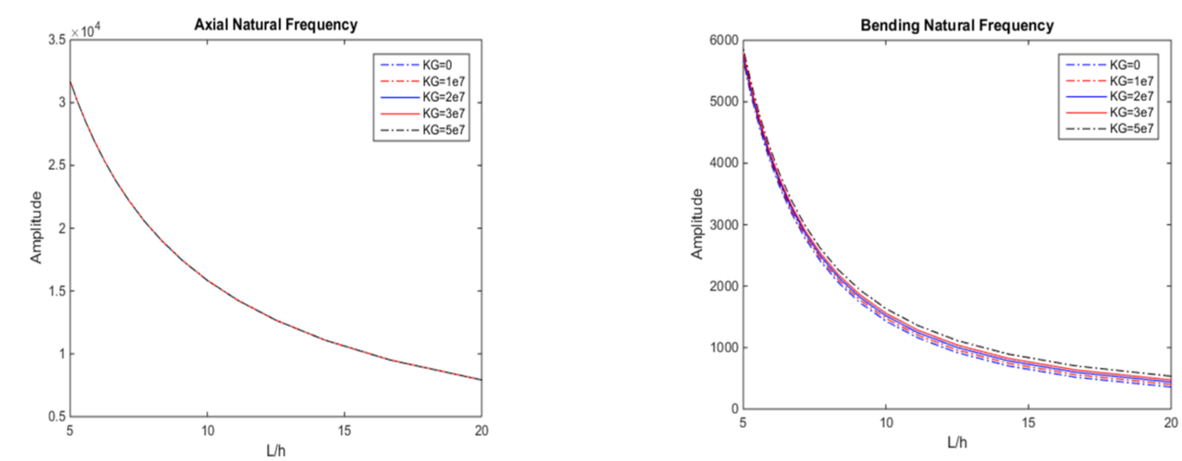
Fig. (5) Natural frequency versus ratio L/h for various Kw. a) Axial natural frequency and b) Bending natural frequency

Fig. (5) Indicates the influence of different spring constant *Kw* on the natural frequencies. In Fig. (5 a), it is observed that the axial natural frequency is not change as the value of the *Kw* change and have the same curve for all values of *Kw*. This is due to that the parameter *Kw* has no effect on axial natural frequency; where it is not involved in the calculation of axial natural frequency equation; as indicated in Eq. (43). In Fig. (5 b) it is noticed that for small values of *Kw* (0, 1e11 and 1e12) the bending natural frequency decreases as the *L/h* ratio increases. For greater values of *Kw* (1e13 and 1e15) the natural frequencies increases as the *L/h* ratio increases (slightly in the case of 1e13 and considerably in the case of 1e15). This is because with increasing the spring parameter (*Kw*), the beam gets stiffer (more rigidity).

It is noticed in Fig. (5 b) that the values of *Kw* are very near in the cases (*Kw*=0, *Kw*=1e11, *Kw*=1e12 and *Kw*=1e13). A sample values are displayed in Table (1) where the values of bending natural frequency (ω_b) are taken for different L/h ratio values. It is observed from Eq. (45) that (ω_b) is proportional to *Kw*. These near values physically are because the stiffness (rigidity) of the beam increase slightly, while in the case of (*Kw*=1e15) the stiffness increase considerably.

Table (1) Near values of bending natural frequency versus ratio *L/h* when changing *Kw* (N/m^3).

L/h	***Kw*=0**	***Kw*=1e11**	***Kw*=1e12**	***Kw*=1e13**	***Kw*=1e15**
5	0.0057e6	0.0125e6	0.0355e6	0.1111e6	1.1099e6
10	0.0014e6	0.0113e6	0.0356e6	0.1123e15	1.1233e6
12.5	0.0009e6	0.0113e6	0.0356e6	0.1125e6	1.1250e6
16.67	0.0005e6	0.0113e6	0.0356e6	0.1126e6	1.1263e6
20	0.0004e6	0.0113e6	0.0356e6	0.1127e6	1.1268e6



(a) (b)

Fig. (6) Natural frequency versus ratio L/h for various KG. a) Axial natural frequency and b) Bending natural frequency

Table (2) Near values of bending natural frequency versus ratio *L/h* when changing K_G (N/m).					
L/h	K_G**=0**	K_G**=1e7**	K_G**=2e7**	K_G**=3e7**	K_G**=5e7**
5	5.6565e3	5.6993e3	5.7418e3	5.7840e3	5.8674e3
10	1.4313e3	1.4742e3	1.5158e3	1.5564e3	1.6344e3
12.5	0.9174e3	0.9600e3	1.0007e3	1.0399e3	1.1141e3
16.67	0.5166e3	0.5585e3	0.5975e3	0.6341e3	0.7016e3
20	0.3589e3	0.4002e3	0.4376e3	0.4720e3	0.5343e3

In Fig. (6), the effect of various shears constant K_G on natural frequencies (axial and bending). In Fig. (6 a), it is observed that the axial natural frequency is not change as the value of the K_G change and have the same curve for all values of K_G. This is due to that the parameter K_G has no effect on axial natural frequency; where it is not involved in the calculation of axial natural frequency equation; as indicated in Eq. (43). In Fig. (6 b) we noticed that the bending natural frequency decreases as the ratio *L/h* increases and the curve take shape of cosine wave. This is because with increasing the spring parameter (K_G), the beam gets stiffer (more rigidity).

It is noticed in Fig. (6 b) that the values of K_G are very near in the cases (K_G=0, K_G=1e7, K_G=2e7, K_G=3e7 and K_G=5e7). A sample values are displayed in Table (2) where the values of bending natural frequency (ω_b) are taken for different L/h ratio values. It is observed from Eq. (42) that (ω_b) is proportional to K_G. These near values physically are because the stiffness (rigidity) of the beam increase slightly.

Figure (7) indicates the axial and bending mode shapes. The parameters considered in this figure are: the length of the beam is proposed to be *L*=1 m, the height h=0.1. As it is noticed in the figure, the axial mode shape takes the shape of cosine wave and the bending mode shape take the shape of sine wave. This due to that the beam material is steel and have high density (7860 kg/m^3) and modulus of elasticity (200 GPa), therefore, all these properties of steel material give the beam high rigidity. Fig. (7) is plotted depending on Eqs. (34) and (35).

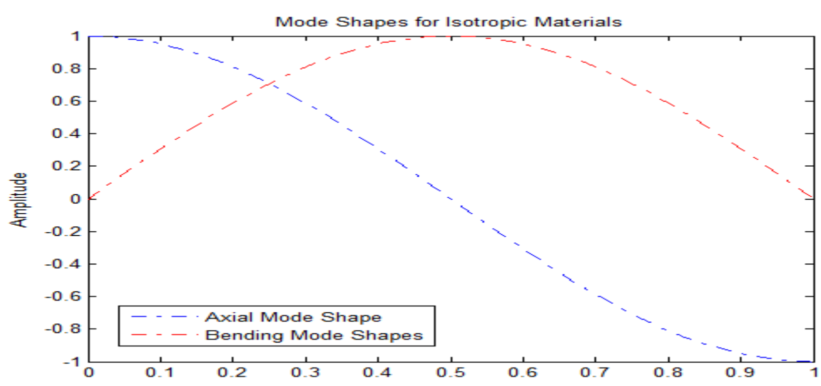
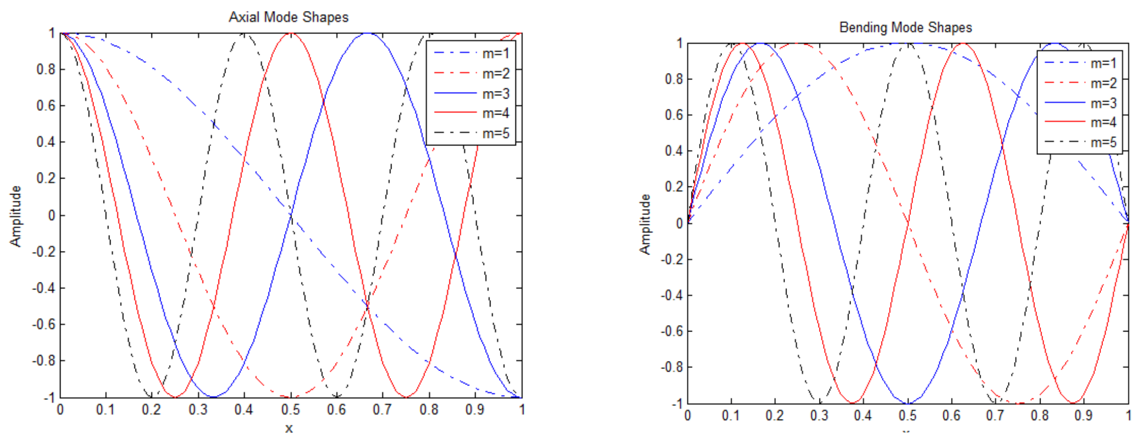


Figure (7) Axial and bending mode shapes versus *L/h*.

In Fig. (8 a) the axial mode shape takes 'cosine-shape'. When changing the parameter (m), 'band width' of wave increases as the parameter (m) decreases. In the other hand, Fig. (8 b) indicates that the bending mode shape takes 'sine-shape'. When changing parameter (m), the 'band-width' of wave increases as parameter (m) decreases. This due to that the beam material is steel and have high density (7860 kg/m^3) and Young modulus of elasticity (200 GPa), therefore, all these properties of steel material give the beam high rigidity. Figure (8 a and b) is plotted depending on Equations (34) and (35), respectively.



(a) (b)

Figure (8) Mode shape versus x when changing m. a) Axial mode shape and b) Bending mode shape

5. Conclusions

In this paper, free vibration of isotropic material Euler-Bernoulli beam resting on Pasternak elastic foundation is considered. The governing equations of motion for isotropic material are derived depending on Hamilton's principle and solved by a numerical method (Navier's method) for simply supported boundary conditions. Navier's method gives accurate and exact solution. The derived formulations were implemented successfully in MATLAB to investigate the effects of changing spring constant (K_w), shear constant (K_G), longitudinal wave number (m), and slenderness ratio (L/h) on natural frequencies (axial natural frequency and bending natural frequency). Natural frequencies were computed for various values of the parameters mentioned using MATLAB. The beam was considered to be made of steel.

The numerical results show that increasing the longitudinal wave number (m) increases both the axial and bending natural frequencies while producing higher-order mode shapes with additional nodal points. The natural frequencies are generally reduced when increasing the slenderness ratio (L/h) because of the reduction in structural stiffness. The spring constant (K_w) notably improves the bending natural frequency, specially at high stiffness values, while it has no effect on the axial natural

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frequency. Also, the shear constant (K_G) influence only the bending frequency. This means that the foundation is what primarily determines the transverse vibration characteristics of the beam.

Generally, the results obtained are consistent with the theoretical behavior of Euler-Bernoulli beams, and prove that the proposed analytical formula provides an accurate and effective tool for predicting the free vibration characteristics of beams resting on Pasternack foundations. The developed formulation can serve as a reliable basis for future research addressing advanced beam theories, Functionally Graded Materials (FGM), nanobeams, and viscoelastic foundation models.

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