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Hyponormality of Toeplitz Operators with Trigonometric Polynomial Symbols

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Abstract

For a trigonometric polynomial φ of the form $\varphi\left(e^{i\theta}\right)=\sum_{n=-M}^{N} a_n\, e^{i n\varphi}$ Where a_{-m} and a_N are nonzero. In this paper, we show that the Toeplitz operator T_φ is hyponormal if and only if $m \le N$ and $|a_{-m}| \le |a_N|$

Keywords: Toeplitz Operators, Hypernormality, Trigonometric Polynomial.

1. Introduction

For φ is a trigonometric polynomial $\varphi\left(e^{i\theta}\right)=\sum_{n=-M}^{N} a_n\, e^{i n\varphi}$, in the present paper we study the hyponormality of T_φ in the cases where φ is a trigonometric polynomial here is to find conditions on the coefficients a_n that are necessary and sufficient for T_φ to be hyponormal.

An elegant and useful theorem of C. Cowen [7] characterizes the hyponormality of Toeplitz operators T_φ on Hardy space $H^2(T)$ of the unit circle $T \subset C$ by properties of the symbol $\varphi \in L^\infty(T)$.

An operator T is said to be hyponnormal if its selfcommutator $\left[T^{*},T\right]=T^{*}T-TT^{*}$ is positive (semidefinte). A. Brown and P. R. Halmos [3] study the relationship between the symbol $\varphi\in L^{\infty}$ and the positivity of the selfcommutator $\left[T^{*},T\right]$.

2. Theorem (1): For each $\varphi\in L^{\infty}$ let

$\mathcal{E}(\varphi)=\left\{k\in H^{\infty}:\|k\|_{\infty}\le 1 \text{ and } \varphi-k\bar{\varphi}\in H^{\infty}\right\}$ A Toeplitz operator T_{φ} is hyponormal if and only if the subset $\mathcal{E}(\varphi)$ of H^{∞} is nonempty.

2.1 Remark (1): Let φ be the trigonometric polynomial $\varphi\left(e^{i\theta}\right)=\sum_{n=-N}^{N}a_n e^{in\varphi}$, where $a_N\neq 0$, and let $k\in H^{\infty}$ satisfies $\varphi-k\bar{\varphi}\in H^{\infty}$, then k necessarily satisfies

$$k\sum_{n=1}^{N}\bar{a}_n e^{-in\theta}-\sum_{n=1}^{N}a_{-n}e^{-in\theta}\in H^{\infty} \qquad (1)$$

(i) The computation of Fourier coefficients $\hat{k}(0),\ldots,\hat{k}(N-1)$ of k is: $\hat{k}(n)=c_n$, for n=0,1, …,N-1, Where $c_0,c_1,\ldots,c_{N-1}$ are determined uniquely from the coefficients of φ by the recurrence relation $c_0=\frac{a_{-N}}{\bar{a}_N}$

$$c_n=\left(\bar{a}_N\right)^{-1}\left(a_{-N+n}-\sum_{j=0}^{n-1}C_j\overline{a_{N-n+j}}\right), \text{ for n=1, …, N-1} \qquad (2)$$

(ii) Therefore if $k_1,k_2\in\mathcal{E}(\varphi)$, then $c_n=\hat{k}_1(n)=\hat{k}_2(n)$ for all $n=0,1,\ldots,N-1$, and $k_p(z)=\sum_{j=0}^{N-1}c_jZ^j$, the unique (analytic) polynomial of degree less than N satisfying $\varphi-k\bar{\varphi}\in H^{\infty}$.

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(iii) Conversely, if k_p is the polynomial $k_p(z)=\sum_{j=0}^{N-1} c_j z^j$, where $c_0, c_1, \ldots c_{N-1}$ are determined from the recurrence relation (2) , then for every integer $n>0$, the Fourier coefficients $\varphi - k\bar{\varphi}(-n)$ of $\varphi - k\bar{\varphi}$ satisfy

$\varphi - k\,\bar{\varphi}(-n) = a_{-n} - \sum_{j=0}^{N-n} c_j \overline{a_{n+j}} = \left(a_{-n} - \sum_{j=0}^{N-n-1} c_j \overline{a_{n+j}} \right) - c_{N-n}\bar{a}_N = 0$, Which implies that $\varphi - k_p\bar{\varphi} \in H^2$. But since $\varphi - k_p\bar{\varphi}$ is a polynomial, it follows that $\varphi - k_p\,\bar{\varphi} \in H^\infty$.

2.2 Remark (2): However the relation (2) can always be solved uniquely to produce an analytic polynomial k_p satisfying $\varphi - k_p\,\bar{\varphi} \in H^\infty$, the polynomial k_p need not be contained in the set $\mathcal{E}(\varphi)$, even if $\mathcal{E}(\varphi)$ is known to be nonempty.

2.3 Example: Consider the trigonometric polynomial $\varphi(e^{i\theta}) = e^{-i2\theta} + 2e^{-i\theta} + e^{i\theta} + 2e^{i2\theta}$

Solving the recurrence relation (2) produces the polynomial $k_p(z) = \frac{1}{2} + \frac{3}{4}z$ which has norm $\|k_p\|_\infty = \frac{5}{4} > 1$ making k_p ineligible for membership in $\mathcal{E}(\varphi)$.

On the other hand, a straight forward calculation shows that the linear fractional transformation $b(z) = \dfrac{z + \frac{1}{2}}{1 + \frac{1}{2}z}$ satisfies $\varphi - b\bar{\varphi} \in H^\infty$, as b maps the unit circle onto itself, b has norm $\|b\|_\infty = 1$. Thus $b \in \mathcal{E}(\varphi)$ and so T_φ is hyponormal. And Fourier series of b, namely $b(e^{i\theta}) \sim \frac{1}{2} + \frac{3}{4}e^{i\theta} - \frac{3}{2}\sum_{j=2}^{\infty}\left(-\frac{1}{2}\right)^n e^{in\theta} = k_p(e^{i\theta}) + h(e^{i\theta})$, Converges uniformly on $(\mathbf{T})$ to b and b is a finite Blaschke product.

Proof. $\varphi\left(e^{i\theta}\right)=\sum_{n=-N}^{N} a_n e^{in\theta}, a_N \neq 0, c_0, c_1, \ldots, c_{N-1} = \; c_o = \frac{a_{-N}}{a_N}$

$$c_n = \left(\bar{a}_N\right)^{-1}\left(a_{-N+N}\sum_{i=n}^{n-1} c_j \bar{a}_{N-n+j}\right), n = 1, \ldots, N-1$$

$$k_p\left(z\right)=\sum_{j=0}^{N-1} c_j z^j = c_0, c_1 \ldots c_{N-1},$$

$$a_{-N}e^{-iN\theta} = e^{-iz\theta} \Rightarrow a_{-N} = \frac{e^{-2i\theta}}{e^{-iN\theta}}, \; a_N e^{iN\theta} = e^{i2\theta} \Rightarrow a_{\;N} = \frac{2e^{i2\theta}}{e^{iN\theta}}, \; \bar{a}_N = \frac{e^{-i2\theta}}{e^{-iN\theta}}$$

$$c_0 = \frac{a_{-N}}{\bar{a}_N} = \frac{e^{-i2\theta}}{e^{-iN\theta}} \times \frac{1}{2}\frac{e^{-iN\theta}}{e^{-i2\theta}} = \frac{1}{2},$$

$$c_1 = \left(\bar{a}_N\right)^{-1}\left(a_{-N+1} - c_0 \bar{a}_{N-1}\right)$$

$$a_{-N+1}e^{i(-N+1)\theta} = 2e^{-i\theta} \Rightarrow a_{-N+1} = \frac{2e^{-i\theta}}{e^{i(-N+1)\theta}}$$

$$a_{-N-1}e^{i(-N+1)\theta} = e^{i\theta} \Rightarrow a_{-N-1} = \frac{e^{i\theta}}{e^{i(-N+1)\theta}}$$

$$c_1 = \frac{1}{2}\frac{e^{i2\theta}}{e^{-iN\theta}}\left[\frac{2e^{-i\theta}}{e^{i(-N+1)\theta}} - \frac{1}{2}\frac{e^{-i\theta}}{e^{i(-N+1)\theta}}\right] = \frac{3}{4}z \quad \left(z = e^{i\theta}\right)$$

$k_p = c_0 + c_1 z = \frac{1}{2} + \frac{3}{4}z$. satisfying $\varphi - k_p \bar{\varphi} \in H^\infty$, where

$$\varphi\left(e^{i\theta}\right) = e^{-i2\theta} + 2e^{-i\theta} + e^{i\theta} + 2e^{i2\theta}$$

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3. Main results

3.1 Theorem (2): suppose that φ is a trigonometric polynomial of the from $\varphi\left(e^{i\theta}\right)=\sum_{n=-m}^{N} a_n e^{in\theta}$, where a_{-m} and a_N are nonzero. If T_φ is hyponormal, then $m \le N$ and $|a_{-m}| \le |a_N|$.

Proof. Suppose T_φ is hyponormal, then φ is trigonometric polynomial under certain assumption about the conefficients $\varphi\left(e^{i\theta}\right)=\sum_{n=-m}^{N} a_n e^{in\theta}$ where $|a_N| \neq 0$, let k salisfies $\varphi - k_p \bar{\varphi} \in H^\infty$ then necessarily salisfies (1), then from (2) $-m \le N$.

$$\varphi\left(e^{i\theta}\right)=\sum_{n=-m}^{N} a_n e^{in\theta} = a_{-m} e^{-im\theta} + a_{-m-1} e^{-i(m+1)\theta} + a_{-m-2} e^{-i(m+2)\theta} + \ldots + a_N e^{iN\theta} a_{-m}$$

Since a_{-m} and a_N are nonzero ,let $c_0, \cdots, c_{N-1}$ be the solution of (2) because $|a_N| \neq 0$, we have $|c_{N-m}| = |a_{-m}| / |a_N|$, then there is a function $k \in \varepsilon(\varphi)$ such that $k(N-m) = c_{N-m}$ thus $1 \ge \|k\|_\infty \ge |c_{n-m}| = |a_{-m}|/|a_N|$ which implies that $|a_{-m}| \le |a_N|$.

3.1.1 Proposition: If $\varphi\left(e^{i\theta}\right)=\sum_{n=-N}^{N} a_n e^{in\theta}$, where $a_N \neq 0$, and if $c_0, c_1, \ldots c_{N-1} \in \mathbb{C}$ are obtained from the coefficients of φ by solving the recurrence relation (2) then the Toeplitz operator T_φ is hyponormal when

$$\sum_{j=0}^{N-1} |c_j| \le 1 \qquad (3)$$

Proof. $c_0 = \frac{a_{-N}}{\overline{a_N}}, c_n = (\bar{a}_N)^{-1}\left(a_{-N+n} - \sum_{j=0}^{n-1} c_j \overline{a_{N-n+j}}\right), n = 1, \ldots, N-1$

$\varphi\left(e^{i\theta}\right)=\sum_{n=-N}^{N} a_n e^{in\theta}$, $k_p(z)=\sum_{j=0}^{N-1} c_j z^j$ satisfies $\varphi-k_p\bar{\varphi}\in H^{\infty}$ from that $\|k_p\|_{\infty}\leq 1$, then $\|k_p\|_{\infty}\leq\sum_{j=0}^{N-1}|c_j|\leq 1$, then $k_p(z)\in\varepsilon(\varphi)$ and so from the Cowen's Theorem T_{φ} is hyponormal.

3.1.2 Remark (1): If $\varphi\left(e^{i\theta}\right)=\sum_{n=-N}^{N} a_n e^{in\theta}$ where $|a_j|\leq|a_N|$, for all $j=2,\dots,N-1$, then from the recurrence relation (2) we have $\sum_{j=0}^{N-1}|c_j|\leq|c_0|+|a_N|^{-2}\sum_{n=1}^{N-1}2^{n-1}|D_n|$, Where $D_n=\det\begin{pmatrix} a_{-n} & a_{-N} \\ \bar{a}_n & \bar{a}_N \end{pmatrix}$. Therefore if $\sum_{n=1}^{N-1}2^{n-1}|D_n|+|a_{-N}\,a_N|\leq|a_N|^2$, (4) Then by proposition (6), T_{φ} is hyponormal. Because the left – hand side of (4) depend on $\overline{a_{-N}}$ and a_N and the right – hand side depends on $|a_N|^2$, it follows that T_{φ} is hyponormal whenever $|a_N|$ is sufficiently large . In particular, the Toeplitz operator with symbol $\varphi+\lambda e^{iN\theta}$ is hyponoarmal whenever $\lambda\in\mathbb{C}$ is such that

$$|\lambda|\geq\sum_{n=1}^{N-1}2^{n-1}\left(|a_{-n}|+|a_n|\right)+|a_{-N}|+|a_N|$$

Proof. Let $\varphi\left(e^{i\theta}\right)=\sum_{n=-N}^{N} a_n e^{in\theta}$ and $|a_j|\leq|a_N|$, $j=2,\dots,N-1$ from (2)

$$\frac{|a_{-N}|}{|\bar{a}_N|}+|a_N|^2\sum_{n=1}^{N-2}2^{n-1}|D_n|$$

$$|D_n|=\left(a_{-n}\bar{a}_N-a_{-N}\bar{a}_n\right)$$

$$c_0=\frac{a_{-N}}{\bar{a}_N}\ \text{(i)}$$

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$c_1 = (\bar{a}_N)^{-1}(a_{-N+1} - c_0 a_{N-1})$ (ii) $\rightarrow c_1 = (\bar{a}_N)^{-1}\left(a_{-N+1} - \frac{a_{-N}}{\bar{a}_N} a_{N-1}\right)$

$c_2 = (\bar{a}_N)^{-1}(a_{-N+2} - c_1 a_{N-2+1})$ (iii) $\rightarrow c_2 = (\bar{a}_N)^{-1}(a_{-N+2} - c_1 a_{N-2+1})$

$c_3 = (\bar{a}_N)^{-1}(a_{-N+3} - c_2 a_{N-3+2})$ (iv) $\rightarrow c_3 = (\bar{a}_N)^{-1}(a_{-N+3} - c_2 a_{N-3+2})$

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$c_n = (\bar{a}_N)^{-1}\left(a_{-N+n} - \sum_{j=0}^{n-1} c_j a_{N-n+j}\right)$ (v)

From (v) $\sum_{j=0}^{n-1} |c_j| \leq 1$

3.1.3 Remark (2): If $a_{-N} = \cdots = a_{-2} = 0$, then the solution to the recurrence relation (2) is $c_0 = \cdots = c_{N-2} = 0$ and $c_{N-1} = a_{-1} / \overline{a_N}$, thus the analytic polynomial.

$k_p \in H^\infty$ is $k_p(z) = (a_{-1} | a_N)^{N-1} z$. Therefore the norm of every $k \in H^\infty$ that satisfies $\varphi - k\bar{\varphi} \in H^\infty$ is such that $\| k \|_\infty \geq \left| \frac{a_{-1}}{\bar{a}_n} \right| = \| k_p \|_\infty$.

Therefore, T_φ is hyponormal if and only if $|a_{-1}| \leq |a_N|$.

The following theorem and corollary concern the extremal cases: $|a_{-m}| = |a_N| \neq 0$

Proof. $a_{-N}=\cdots=a_{-2}=0$. then the solution is $c_0=\cdots=c_{N-2}=0$ $c_{N-1}=\frac{a_{-1}}{a_N}$, thus $k_p\in H^\infty=\sum_{j=0}^{N-1}c_jz^j$, $c_0=\frac{a_{-N}}{\bar{a}_N}$ since $c_{N-m}=\left|\frac{a_{-m}}{\bar{a}_N}\right|$ implies that $c_{N-1}=\left|\frac{a_{-1}}{\bar{a}_N}\right|, c_{N-2}=\left|\frac{a_{-N-2}}{\bar{a}_N}\right|$ implies $\sum_{j=0}^{N-2}|c_j|\le 1$ from the proposition (6) and $\|k_p\|_\infty\le\sum_{j=0}^{N-1}|c_j|\le 1$, since ,$1\ge\|k\|_\infty\ge\|c_{N-n}\|$ thus $\|k\|_\infty\ge\left|\frac{a_{-1}}{\bar{a}_N}\right|=\|k_p\|_\infty$.

Therefor, T_φ is hyponormal if and only if $|a_{-1}|\le|a_N|$.

3.2 Theorem (3): There exists a finite Blaschke product $b\in\mathcal{E}(\varphi)$ of degree equal to the rank of $[T_\varphi^*,T_\varphi]$.

3.3 Theorem (4): Suppose that $\varphi(e^{i\theta})=\sum_{n=m}^{N}a_n e^{in\theta}$, where $m\le N$ and $|a_{-m}|\le|a_N|\ne 0$, and let $\mathcal{E}(\varphi)\subset H^\infty$ be the subset of all $k\in H^\infty$ for which $\|k\|_\infty\le 1$ and $\varphi-k\bar{\varphi}\in H^\infty$. The following statements are equivalent.

(a) The Toeplitz operator T_φ is hyponormal.

(b) For all $k=1,\dots,N-1$, $\det\begin{pmatrix}\frac{a_{-(m-k)}}{a_{N-m+1}} & \frac{a_{-m}}{a_N}\end{pmatrix}=0$.

(c) The following equation in $\mathbb{C}^m$ holds:

$$\overline{a_N}\begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ \vdots \\ a_{-m} \end{pmatrix} = a_{-m}\begin{pmatrix} \overline{a_{N-m+1}} \\ \overline{a_{N-m+2}} \\ \vdots \\ \vdots \\ \overline{a_N} \end{pmatrix} \quad (5)$$

(d) $\mathcal{E}(\varphi)=\left\{a_{-m}\left(\overline{a_N}\right)^{-1} z^{N-m}\right\}$.

Moreover, if T_φ is hyponormal , then the rank of $\left[T_\varphi^*, T_\varphi\right]$ is $N-m$.

Proof. Let $c_0,\cdots,c_{N-1}$ be the solution to (2) ; because $|a_{-m}|=|a_N|\neq 0$, we have $|c_{N-m}|=1$. Note that if $m<N$, then $c_0=\cdots=c_{N-m-1}=0$. If a function $k\in H^\infty$ satisfies $\varphi-k\bar{\varphi}\in H^\infty$, then the Fourier series expansion of k is

$k=\sum_{j=0}^{N-1} c_j e^{ij\theta}+\sum_{n=N}^{\infty} b_n e^{in\theta}$ for some set of $b_n\in\mathbb{C}$.

From fact $\|k\|_\infty \geq \|k\|_2$ we have $\|k\|_\infty \geq |c_{N-m}|=1$; if for some $j>(N-m)$ or $n\geq N$ there is a nonzero Fourier coefficient c_j or b_n of k, then $\|k\|_\infty \geq \max\left\{\sqrt{|c_{N-m}|^2+|c_j|^2}, \sqrt{|c_{N-m}|^2+|b_n|^2}\right\}>1$.

Thus $\|k\|_\infty=1$ if and only if c_{N-m} is the only nonzero Fourier coefficient of k . Therefore $\mathcal{E}(\varphi)$ can have at most one element : namely $c_{N-m}z^{N-m}$ Hence, statements (a) and (d) are equivalent. Now statement (a) and (b) are equivalene; clearly (b) and (c) are exact same statement. Suppose that T_φ is hyponormal. Then there exists $k\in\mathcal{E}(\varphi)$ and $k(z)=c_{N-m}z^{N-m}$. Hence , for every $k=1,\cdots,m-1$,

$$0=\left|c_{N-m+k}\right|=\left|\frac{1}{\overline{a_N}}\left(a_{-(m-n)}\right)-c_{N-m}\,\overline{a_{N-k}}\right|=\left|\frac{1}{\overline{a_N}}\right|^2\left|\det\begin{pmatrix}a_{-(m-k)} & a_{-m}\\ \overline{a_{(N-k)}} & \overline{a_N}\end{pmatrix}\right|.$$

Conversely, if $\det\begin{pmatrix}a_{-(m-k)} & a_{-m}\\ \overline{a_{(N-k)}} & \overline{a_N}\end{pmatrix}=0$ for all $k=1,\cdots,N-1$ then

$$\left|c_{N-m+1}\right|=\left|\frac{1}{\overline{a_N}}\left(a_{-(m-1)}-c_{N-m}\,\overline{a_{N-1}}\right)\right|=\left|\frac{1}{\overline{a}_N}\right|^2\det\begin{pmatrix}a_{-(m-k)} & a_{-m}\\ \overline{a_{(N-1)}} & \overline{a}_N\end{pmatrix}=0.$$

and hence $\left|c_{N-m+2}\right|=\left|\frac{1}{\overline{a_N}}\left(a_{-(m-2)}-c_{N-m}\,\overline{a_{N-2}}-c_{N-m-1}\,\overline{a_{N-1}}\right)\right|=\left|\frac{1}{\overline{a_N}}\right|^2\det\begin{pmatrix}a_{-(m-2)} & a_{-m}\\ \overline{a_{(N-2)}} & \overline{a_N}\end{pmatrix}=0.$

Inductively, we obtain $c_k=0$ for all k=1,…,N-1 . As $c_0=\cdots=c_{N-m-1}=0$, if $m<N$ and $\left|c_{N-m}\right|=1$, we have, that the analytic polynomial $k_p(z)=\sum_{j=0}^{N-1}$ is of the form $k_p(z)=c_{N-m}z^{N-m}$ and therefore $k_p\in\mathcal{E}(\varphi)$. This completes the proof that statements (i) and (ii) are equivalent.

Lastly, if T_φ is hyponormal, then $\mathcal{E}(\varphi)=\left\{\frac{a_{-m}}{\overline{a}_N}z^{N-m}\right\}$. Because the self commentator $\left[T_\varphi^*,T_\varphi\right]$ has finite rank, ([18], theorem 10), there is only one element in $\mathcal{E}(\varphi):b(z)=\frac{a_m}{\overline{a}_N}z^{N-m}$, which is a finite Blaschke product of degree N-m.

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3.4 Corollary: suppose that $\varphi\left(e^{i\theta}\right)=\sum_{n=-m}^{N} a_n e^{in\theta}$, where $m \le N$ and $|a_{-m}|=|a_{-N}|\ne 0$ and let $\varepsilon(\varphi)\subset H^{\infty}$ be the subset of all $k\in H^{\infty}$ for which $\|k\|_{\infty}\le 1$ and $\varphi-k\bar{\varphi}\in H^{\infty}$.the following statements are equivalent :

(a) The Toeplitz operator T_φ is hyponormal .

(b) For all $k=1,\dots,N-1$, $\det\begin{pmatrix} \frac{a_{-(m-k)}}{a_{N-m+1}} & \frac{a_{-m}}{a_N} \end{pmatrix}=0$.

Proof. Suppose T_φ is hyponormal from remark (8) the analytic polynomial (ii) holds for all $k=1,\dots,N-1$. For backward implication since $|\bar{a}_N|\ne 0$ and $|\bar{a}_{-m}|\ne 0$,

$$\det\begin{pmatrix} \frac{a_{-(m-k)}}{a_{(N-k)}} & \frac{a_{-m}}{\bar{a}_N} \end{pmatrix}=a_{-(m-k)}\bar{a}_N-a_{-m}\bar{a}_{N-k}=a_{-m+k}\bar{a}_N-a_{-m}a_{N-k}$$

$$=a_{-m+k}\bar{a}_N-a_{-m+k}\bar{a}=0$$

Then from proposition (6) and the remarks (7) and (8) T_φ is hyponormal

3.5 Example: T_φ is hyponormal with rank-2self commutator rank $\left[T_{\varphi_2}^*, T_{\varphi_2}\right]=2$, $\varphi_2\left(e^{i\theta}\right)=e^{-i2\theta}+e^{-i\theta}+e^{i3\theta}+e^{i4\theta}$

Prove that T_{φ_2} is hyponormal with rank-2self commutator.

Proof. Let $c_0=c_1=\dots,c_{N-m-1}=0$, $k:1=1,\dots,N-1,\det\begin{pmatrix} \frac{a_{-(m-n)}}{a_{(N-n)}} & \frac{a_{-m}}{\bar{a}_N} \end{pmatrix}=0$, $\varphi\left(e^{i\theta}\right)=\sum_{n=-m}^{N} a_n e^{in\theta}$,

$$c_0=\frac{a_{-m}}{\bar{a}_N},\quad c_{-m-1}=\frac{a_{-m-1}}{\bar{a}_N}\quad,\quad \frac{a_{-m-2}}{\bar{a}_N}=c_{-m-2}\dots c_{-m-(N-1)}=\frac{a_{-m-(N-1)}}{\bar{a}_N}$$

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Then $\begin{pmatrix} e^{-i\theta} & e^{-2i\theta} \\ e^{-i3\theta} & e^{-i4\theta} \end{pmatrix} = e^{-5i\theta} - e^{-5i\theta} = 0$

Then T_{φ_2} is hyponormal.

3.6 Example: Applied thereom (10) to show that the Toeplitz operator with symobol

$\varphi\left(e^{i\theta}\right) = e^{-i5\theta} - e^{-i4\theta} + e^{-i2\theta} + e^{-i\theta} + 2e^{i2\theta} - 2e^{i4\theta} + 2e^{i5\theta}$

Whose coefficients satisfy the symmetric relation

$$\overline{a_N}\begin{pmatrix} a_{-2} \\ a_{-3} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N}\begin{pmatrix} \bar{a}_{-2} \\ \bar{a}_{-3} \\ \vdots \\ \bar{a}_{-N} \end{pmatrix}$$

But for which there is no symmetry involving a_{-1} and a_1 is hyponormal .

Proof. $\varphi\left(e^{i\theta}\right) = \sum_{n=-N}^{N} a_n e^{in\theta}$

$$a_N e^{iN\theta} = 2e^{i5\theta}$$
$$\bar{a}_N = 2e^{-i5\theta} / e^{-iN\theta}$$

$$\bar{a}_1 = 2e^{-i5\theta} / e^{-i\theta}$$
$$\bar{a}_2 = 2e^{-i5\theta} / e^{i2\theta}$$
$$\bar{a}_3 = 2e^{-i5\theta} / e^{-i3\theta}$$

$$a_{-N} e^{-iN\theta} = e^{-i5\theta} \Rightarrow a_{-N} = e^{-i5\theta} / e^{-iN\theta}$$

$$a_{-1} = \frac{e^{-i5\theta}}{e^{i\theta}},\ a_{-2} = \frac{e^{-i5\theta}}{e^{i2\theta}}\ ,a_{-3} = \frac{e^{-i5\theta}}{e^{i3\theta}}$$

$$2e^{-i5\theta}/e^{-iN\theta}\begin{pmatrix} e^{-i5\theta}/e^{i2\theta} \\ e^{i5\theta}/e^{i3\theta} \\ \vdots \\ e^{-i5\theta}/e^{-iN\theta} \end{pmatrix} = e^{-i5\theta}/e^{-iN\theta}\begin{pmatrix} 2e^{i5\theta}/e^{-i2\theta} \\ 2e^{-i5\theta}/e^{-i3\theta} \\ \vdots \\ 2e^{-i5\theta}/e^{-iN\theta} \end{pmatrix}$$

3.7 Corollary: If $\varphi(e^{i\theta}) = \sum_{n=-m}^{N} a_n e^{in\theta}$, then T_φ is normal if and only if $m = N$, $|a_{-N}| = |a_N|$

and $$\overline{a_N}\begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N}\begin{pmatrix} \overline{a_1} \\ \overline{a_2} \\ \vdots \\ \vdots \\ \overline{a_N} \end{pmatrix} \qquad (6)$$

Proof. If $m = N$, $|a_{-m}| = |a_N|$, and let $\det\begin{pmatrix} a_{-(m-k)} & a_{-m} \\ \overline{a_{(N-k)}} & \overline{a_N} \end{pmatrix} = 0$ for all $k = 1, \ldots, N-1$, then by Theorem (11), T_φ is hyponormal and rank $[T_\varphi^*, T_\varphi] = N - m = o$; that is T_φ normal. Conversely, if T_φ is normal, then by [Brown – Halmos [13]], there are scalars $\alpha, \beta \in \mathbb{C}$ and areal –value $\psi \in L^\infty$ such that $T_\varphi = \alpha T_\psi + \beta 1$. As T_ψ is a hermitian Toeplitz operator , the Fourier coefficients of ψ satisfy $\hat{\psi}(n) = \overline{\hat{\psi}(-n)}$ for all n ; in particular $|\alpha||a_N| = |\hat{\psi}(N)| = |\hat{\psi}(-N)| = |\alpha||a_{-N}|$, Showing that $|a_{-N}| = |a_N|$. Thus, N=m and (16) holds.

3.8 Remark: For trigonometric polynomials φ satisfying the assumptions of theorem (11) the question of whether or not the Toeplitz operator T_φ is hyponormal is completely independent of the values the coefficients $a_0, \cdots, a_{N-m}$ of φ.

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