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Families of Disjoint Sets Colouring (Partitioning) Technique and Standard of Partition for Edge Colouring

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Abstract

Families of disjoint sets colouring (partitioning) technique is trial to unify and generalize all types of colouring such as edge colouring, vertex colouring, and face colouring. The concept of standard of partitioning edges into colour classes is essential concept to follow and understand families of disjoint sets colouring (partitioning) technique. In this paper, we introduce some examples of standard of partitioning, and introduce some results of partitioning edges. Some of these results explain importance of standard of partitioning related to subgraphs, and some results related to maximum number of edges belongs to each colour class. Also, we introduce results of partitioning edges related to quasi non common edges. Here we list some properties related to standard of partitioning edges into colour classes.

Keywords: Minimum Colour Classes, Standard of Partitioning Edges.

I. Introduction

In paper [4], we introduce families of disjoint sets colouring technique as trial to unify and generalize all types of colouring such as edge colouring, vertex colouring, face colouring and set colouring (set partitioning). Concept of common object is essential concept for families of disjoint sets. In papers [12], [13], [14] and [16] we introduce the concept of common set, common edge, common vertex and common face respectively. In papers [5], [6], [7], [8], [9], [10] and [11] we introduce some concepts which had been modified later in papers [12], [13], [14] and [15], such as minimum number of colour classes, maximum number of colour classes, and standard of partition.

In section three of this paper, we introduce some examples of standard of partitioning edges into colour classes. In section four we introduce some results comparison between standard of partitioning edges of a graph and standard of partitioning edges of its subgraphs. In this section also we introduce concept of quasi non-common edge, and some results related to this concept and concept of standard of partitioning edges. In section five we introduce some results explain some properties of standard of partitioning edges into colour classes, properties related to maximum degree of vertex. In section six we introduce some results about relation between maximum number of edges belongs to each colour class, and standard of partitioning edges into minimum number of colour classes. In section seven we introduce some results partitioning edges into minimum number of colour classes, when vertices of graph satisfy certain conditions.

II. Related Work

Definitions and results in this section will be needed through this paper.

Proposition 2.1. Let G be a graph of k edges $e_1, e_2, e_3, ..., e_k$ and $v_1, v_2, v_3, ..., v_w$ are proper vertices of these k edges, with maximum degree of vertex equal $n,$ where $2 < n,$ and there is vertex of degree equal two. If all edges (common edges and non-common edges) partitioned into m colour classes, then the number of colour classes appear w times at $v_1, v_2, v_3, ..., v_w$ not more than two [13].

Definition 2.2. Let G be a graph of k common and non-common edges $e_1, e_2, e_3, ..., e_k$ (common between proper vertices $v_1, v_2, v_3, ..., v_w$), let these k edges partition into m colour classes, then these m colour classes are called minimum colour classes, if whatever we try to partition these k edges into $m-1$ colour classes, there exist at least two adjacent edges belong to the same colour class[15].

Definition 2.3. Let G be a graph of common and non-common edges $e_1, e_2, e_3, ..., e_k$ (common between proper vertices $v_1, v_2, v_3, ..., v_w$), these k edges partition into m colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, if w^* be minimum number of proper vertices $v_1^*, v_2^* v_3^*, ..., v_{w^*}^*$ of common edges $e_1^*, e_2^*, e_3^*, ..., e_{k^*}^*$, satisfies $w^* + 1 \le t_i + t_j$, for any two colour classes χ_i and χ_j, then the vertices $v_1^*, v_2^* v_3^*, ..., v_{w^*}^*$ called standard of partition these k^* edges into minimum m colour classes, where t_i and t_j be number of times the colour classes χ_i and χ_j appears at these w^* respectively, where $k^* \le k$, $w^* \le w$, $V^* \subseteq V$, $E^* \subseteq E$, $E = \{e_1, e_2, e_3, ..., e_k\}$, $E^* = \{e_1^*, e_2^*, e_3^*, ..., e_{k^*}^*\}$, $V^* = \{v_1^*, v_2^*, v_3^*, ..., v_{w^*}^*\}$, $V = \{v_1, v_2, v_3, ..., v_w\}$, $1 \le i < j \le m.$ These m colour classes are called maximum colour classes, if whatever we try to partition these k edges into $m+1$ colour classes, there exist two colour classes χ_i and χ_j such that $t_i + t_j \le w^*$, where $1 \le i < j \le m+1$ [15].

Remark: 2.4. By proper vertex we mean vertex with degree equal one, when we deal with edge colouring. From definition of clique and proper clique there is no clique with clique number equal one, when we deal with vertex colouring. From definition of intersection and proper intersection there is no intersection with degree of intersection equal one, when we deal with set partitioning.

Proposition 2.5. Suppose we have common edges $e_1, e_2, e_3, ..., e_k$ and $v_1, v_2, v_3, ..., v_w$ are proper vertices of these k edges, with maximum degree of vertex equal n, suppose all edges (common edges and non-common edges) partition into m colour classes

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$\chi_1, \chi_2, \chi_3, ..., \chi_m$, for any two colour classes χ_i meet χ_j at these w vertices, if t_i, t_j be number of times the colour classes χ_i, χ_j appear at these w vertices respectively, where these w vertices is standard of partition of all edges (common edges and non-common edges) into m colour classes, then we have

(i) $w+1 \le t_i + t_j$ for w odd or even. (ii) $\frac{w+1}{2} \le t_i \le w$ (If there is special case existence of colour class χ_i such that $2 \le t_i \le \frac{w-1}{2}$ it will be only one colour class) for w odd, where $1 \le i < j \le m$.

(iii) $\frac{w+2}{2} \le t_i \le w$ (If there is special case existence of colour class χ_i such that $2 \le t_i \le \frac{w}{2}$ it will be only one colour class) for w even, where $1 \le i < j \le m$. (iv) If there exist two colour classes χ_i, χ_j such that $t_i \le t_j \le \frac{w-1}{2}$ for w odd or $t_i \le t_j \le \frac{w}{2}$ for w even, then χ_i and χ_j they will form only one colour class[14].

Proposition 2.6. Suppose we have common edges $e_1, e_2, e_3, ..., e_k$ and $v_1, v_2, v_3, ..., v_w$ are proper vertices of these k edges, with maximum degree of vertex equal n, suppose these k edges and non-common edges partitioned into m colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, labelled by $1,2,3,...,m$, and if t_i be number of times the colour class χ_i appear at these w vertices, and for all $1 \le i \le m$, $1 \le l \le k$, $1 \le j \le w$ and k_1 is number of non-common edges, then we have $\sum_{i=1}^{m} t_i = k_1 + \sum_{i=1}^{k} w_i = k_1 + \sum_{i=1}^{k} c_i = k_1 + 2k = \sum_{j=1}^{w} d(v_j)$. [13].

Proposition 2.7. Let G be a graph of k common edges $e_1, e_2, e_3, ..., e_k$, these k common edges intersect at vertices $v_1, v_2, v_3, ..., v_w$, with maximum degree of vertex equal n, and be partitioned into m minimum colour classes, if we add only one non common sets, without changing maximum degree of vertices, and without addition of any proper vertex, only degree of one proper vertex changes, then these k common edges and non-common edges can be partitioned into m minimum colour classes [13].

Proposition 2.8. Let G be a graph of k common edges $e_1, e_2, e_3, ..., e_k$, these k common edges intersect at vertices $v_1, v_2, v_3, ..., v_w$, with maximum degree of vertex equal n, these edges (common and non-common edges) can be partitioned into m colour classes, if we remove only one non common edge, without changing maximum degree of vertex, and without removal of any proper vertex of $v_1, v_2, v_3, ..., v_w$, then after this removal, common edges and non-common edges partitioned into m minimum colour classes [13].

Theorem 2.9. The sum of degrees of vertices equals twice the number of edges [1].

Definition 2.10. A clique of a graph is a maximal complete sub graph. A clique number n of a graph G is largest number such that is a subgraph K_n of G [14].

Iii. Some Examples for Standard of Partition

In this section we introduce some examples of standard of partition edges into colour classes.

Remark 3.1. Let G be a graph of k edges $e_1, e_2, e_3, ..., e_k$ and these k edges intersect at w proper vertices $v_1, v_2, v_3, ..., v_w$, with maximum degree of vertex equal n, if theses k edges $e_1, e_2, e_3, ..., e_k$ partitioned into m minimum colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, labelled by $1, 2, 3, ..., m$ respectively. Also, theses k edges can be labelled by $1, 2, 3, ..., k$ respectively. So, we have two types of labelling. First labelling a vertex and an edge by labels of colour classes, second labelling a vertex and an edge by labels of edges.

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Example 3.2. Let G be a graph of three common edges e_1, e_2, e_3 and three vertices v_1, v_2, v_3, all vertices of degree two. These edges are partitioned into three minimum colour classes χ_1, χ_2, χ_3, each colour class consists of one edge.

The edge e_1 is common between v_1 and v_2, so we can used the symbol $e_1 \equiv (v_1, v_2)$, or the symbol $\{e_1\} = \{v_1, v_2\}$.

$\chi_1 = \{e_1\} = \{v_1, v_2\}$.

$\chi_2 = \{e_2\} = \{v_2, v_3\}$.

$\chi_3 = \{e_3\} = \{v_3, v_1\}$.

Where t_1 is appearance of colour class χ_1 at w vertices v_1, v_2, v_3 is satisfy definition of standard of partition $w = 3,\ t_1 = t_2 = t_3 = 2$, where $w + 1 \le t_i + t_j,\ t_1$ is appearance of colour class χ_1 at w vertices and $1 \le i < j \le 3$.

Example 3.3. Let G be a graph of five non-common edges e_1, e_2, e_3, e_4, e_5 and six vertices $v_1, v_2, v_3, ..., v_6$, , all vertices of degree one $d(v_2) = d(v_3) = d(v_4) = d(v_5) = d(v_6) = 1$, except one vertex $v_1,\ d(v_1) = 5$, these edges partitioned into five colour classes $\chi_1, \chi_2, \chi_3, \chi_4, \chi_5$. Standard of partition consists of one vertex of standard of partition v_1.

Remark 3.4. Let G be a graph of k common edges and $v_1, v_2, v_3, ..., v_w$ are common vertices of these k edges, with maximum degree of vertex equal n, if these k edges partitioned into m minimum colour classes, where $m = n$, then any vertex of degree n is standard of partition, see Example 3.2. And Example 3.3.

Example 3 .5. Let G be a graph of one common edge e_1, e_1 is common between v_1 and v_{10}, eight non common edges $e_2, e_3, e_4, ..., e_9$, and ten vertices $v_1, v_2, v_3, ..., v_{10}$, $d(v_1) = d(v_{10}) = 5,\ d(v_2) = d(v_3) = d(v_4) = d(v_5) = d(v_6) = d(v_7) = d(v_8) = d(v_9) = 1$. The edge e_1 is

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common between v_1 and v_{10}, so we can used the symbol $e_1 \equiv (v_1, v_{10})$, or the symbol $\{e_1\} = \{v_1, v_{10}\}$, $\{e_2\} = \{v_1, v_2\}$, $\{e_3\} = \{v_1, v_3\}$, $\{e_4\} = \{v_1, v_4\}$, $\{e_5\} = \{v_1, v_5\}$, $\{e_6\} = \{v_{10}, v_6\}$, $\{e_7\} = \{v_{10}, v_7\}$, $\{e_8\} = \{v_{10}, v_8\}$, $\{e_9\} = \{v_{10}, v_9\}$. These nine edges partitioned into five colour classes $\chi_1, \chi_2, \chi_3, \chi_4, \chi_5$, as follows:

$\chi_1 = \{e_1\} = \{v_1, v_{10}\}$,

$\chi_2 = \{e_2, e_6\} = \{v_1, v_2, v_6, v_{10}\}$.

$\chi_3 = \{e_3, e_7\} = \{v_1, v_3, v_7, v_{10}\}$.

$\chi_4 = \{e_4, e_8\} = \{v_1, v_4, v_8, v_{10}\}$.

$\chi_5 = \{e_5, e_9\} = \{v_1, v_5, v_9, v_{10}\}$.

We have three standards of partition consists $\{v_1\}$, $\{v_{10}\}$ and $\{v_1, v_{10}\}$. Let $\{v_1\}$ be standard of partition, then we have $t_1 = 1, t_2 = 1, t_3 = 1, t_4 = 1, t_5 = 1.$ Let $\{v_{10}\}$ be standard of partition, then we have $t_1 = 1, t_2 = 1, t_3 = 1, t_4 = 1, t_5 = 1.$ Let $\{v_1, v_{10}\}$ be standard of partition, then we have $t_1 = 2, t_2 = 2, t_3 = 2, t_4 = 2, t_5 = 2.$ In case standard of partition $\{v_1\}$ we have $w = 1$, in case standard of partition $\{v_{10}\}$ we have $w = 1$, and in case standard of partition $\{v_1, v_{10}\}$ we have $w = 2$, in each case we have $w + 1 \le t_i + t_j$, where $1 \le i < j \le 5$.

Example 3.6. Let G be a graph of five common edges e_1, e_2, e_3, e_4, e_5 and three vertices $v_1, v_2, v_3,$, $d(v_1) = 4$ and $d(v_2) = d(v_3) = 3.$ The edges e_1, e_2 are multiple edges both common between v_1, v_2. The edges e_3, e_4 are multiple edges both common between v_1, v_3. The edge e_5 is common between v_2, v_3. These five edges partitioned into five minimum colour classes $\chi_1, \chi_2, \chi_3, \chi_4, \chi_5$, each colour class consists of one edge.

$\chi_1 = \{e_1\} = \{v_1, v_2\}$.

$\chi_2 = \{e_2\} = \{v_1, v_2\}$.

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$\chi_3 = \{e_3\} = \{v_1, v_3\}.$

$\chi_4 = \{e_4\} = \{v_1, v_3\}.$

$\chi_5 = \{e_5\} = \{v_2, v_3\}.$

$t_1 = t_2 = t_3 = t_4 = t_5 = 2,$ Where t_i is appearance of colour class χ_i at w vertices $v_1, v_2, v_3,$ $w = 3,$ and $1 \le i < j \le 5.$

The vertices v_1, v_2, v_3 satisfies definition of standard of partition where $w+1 \le t_i + t_j.$

Example 3.7. Let G be a graph of seven common edges and five vertices $v_1, v_2, v_3, v_4, v_5,$ all vertices of degree three and one vertex of degree two. These seven edges partitioned into four minimum colour classes $\chi_1, \chi_2, \chi_3, \chi_4,$ each colour class of χ_1, χ_2, χ_3 consists of two edges, the class χ_4 consists of only one edge, this common edge belongs alone to the colour class χ_4. We used the following symbol $\chi_1 = \{e_1, e_3\},$ $\chi_2 = \{e_2, e_4\},$ $\chi_3 = \{e_5, e_6\},$ $\chi_4 = \{e_7\}.$ The edge e_1 is common between v_1 and v_2, so we used the symbol $e_1 = \{v_1, v_2\},$ or symbol $e_1 \equiv (v_1, v_2).$ We continue with same labels and we write:

$\chi_1 = \{e_1, e_3\} = \{(v_1, v_2), (v_3, v_4)\} = \{v_1, v_2, v_3, v_4\}.$

$\chi_2 = \{e_2, e_4\} = \{(v_2, v_3), (v_4, v_5)\} = \{v_2, v_3, v_4, v_5\}.$

$\chi_3 = \{e_5, e_6\} = \{(v_1, v_5), (v_2, v_4)\} = \{v_1, v_5, v_2, v_4\}.$

$\chi_4 = \{e_7\} = \{(v_3, v_5)\} = \{v_3, v_5\}.$

$t_1 = t_2 = t_3 = 4,$ $t_4 = 2,$ Where t_i is appearance of colour class χ_i at w vertices $v_1, v_2, v_3, v_4, v_5,$ $w = 5,$ and $1 \le i < j \le 4.$ the vertices $v_1, v_2, v_3, v_4, v_5,$ satisfies definition of standard of partition where $w+1 \le t_i + t_j.$

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Example 3.8. Let G_1 be a graph of seven common edges and five vertices $v_1, v_2, v_3, v_4, v_5,$ all vertices of degree three and one vertex of degree two. These seven edges partition into four minimum colour classes $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4,$ each colour class of $\Gamma_1, \Gamma_2, \Gamma_3$ consists of two edges, the class Γ_4 consists of only one edge, this common edge belongs alone to the colour class Γ_4. We used the following symbol $\Gamma_1 = \{e_1, e_3\}$, $\Gamma_2 = \{e_2, e_4\}$, $\Gamma_3 = \{e_5, e_6\}$, $\Gamma_4 = \{e_7\}$.

$\Gamma_1 = \{e_1, e_3\} = \{(v_5, v_1), (v_2, v_3)\} = \{v_5, v_1, v_2, v_3\}.$

$\Gamma_2 = \{e_2, e_4\} = \{(v_1, v_2), (v_3, v_4)\} = \{v_1, v_2, v_3, v_4\}.$

$\Gamma_3 = \{e_5, e_6\} = \{(v_5, v_4), (v_1, v_3)\} = \{v_5, v_4, v_1, v_3\}.$

$\Gamma_4 = \{e_7\} = \{(v_2, v_4)\} = \{v_2, v_4\}.$

Let G_2 be a graph of three common edges $e_8, e_9, e_{10},$ and four non common edges $e_{10}, e_{11}, e_{12}, e_{13},$ four proper vertices $v_5, v_6, v_7, v_8,$ and four non proper vertices $v_9, v_{10}, v_{12}, v_{13},$ where e_8 is common edge between v_5 and v_6. e_9 is common edge between v_6 and v_7, e_{10} is common edge between v_7 and v_8, e_{11} is non-common edge between v_7 and v_9, e_{12} is non-common edge between v_7 and v_{10}, e_{13} is non-common edge between v_8 and v_{11}, e_{14} is non-common edge between v_8 and v_{12}. The seven edges of G_2 partitioned into three minimum colour classes $\Psi_1, \Psi_2, \Psi_3,$ We join v_5 in G_1 to v_5 in G_2, then we have $d(v_1) = d(v_2) = d(v_3) = d(v_4) = d(v_5) = d(v_6) = d(v_7) = d(v_8) = 3,$ and
$d(v_9) = d(v_{10}) = d(v_{11}) = d(v_{12}) = 1.$

Common between v_1 and v_2, so we used the symbol $e_1 = \{v_1, v_2\}$, or symbol $e_1 \equiv (v_1, v_2)$. we continue with same notations and we write:

$\Psi_1 = \{e_9, e_{14}\} = \{(v_6, v_7), (v_8, v_{12})\} = \{v_6, v_7, v_8, v_{12}\}.$

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$\Psi_2 = \{e_8, e_{11}, e_{13}\} = \{(v_5, v_6), (v_7, v_9), (v_8, v_{11})\} = \{v_5, v_6, v_7, v_9, v_8, v_{11}\}.$

$\Psi_3 = \{e_{10}, e_{12}\} = \{(v_6, v_8), (v_7, v_{10})\} = \{v_6, v_8, v_7, v_{10}\}.$

Let $G = G_1 \cup G_2$ and all edges partitioned into four colour classes $\chi_1, \chi_2, \chi_3, \chi_4$, where $\chi_1 = \Gamma_1 \cup \Psi_1$, $\chi_2 = \Gamma_2 \cup \Psi_2$, $\chi_3 = \Gamma_3 \cup \Psi_3$, and $\chi_4 = \Gamma_4$. If we consider these eight proper vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$ standard of partition, then we have $w = 8$, and $t_1 = t_3 = 7, t_2 = 8$, $t_4 = 2$, then these eight proper vertices satisfy condition of standard of partition, $w + 1 \le t_i + t_j$, where $1 \le i < j \le 4$.

Example 3.9. Let G be a graph of twelve edges, eleven common edges and one edge is non-common edge, ten vertices, $d(v_1) = d(v_2) = d(v_3) = d(v_4) = d(v_5) = 3$, $d(v_6) = d(v_7) = d(v_8) = d(v_9) = 2$, and $d(v_{10}) = 1$. we partition all edges into four minimum colour classes $\chi_1, \chi_2, \chi_3, \chi_4$, as follows:

$\chi_1 = \{e_1, e_3, e_9, e_{11}\} = \{(v_5, v_1), (v_2, v_3), (v_6, v_7), (v_8, v_9)\} = \{v_5, v_1, v_2, v_3, v_6, v_7, v_8, v_9\}.$

$\chi_2 = \{e_2, e_4, e_8, e_{10}, e_{12}\} = \{(v_1, v_2), (v_3, v_4), (v_5, v_6), (v_7, v_8), (v_9, v_{10})\} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v, v\}.$

$\chi_3 = \{e_5, e_6\} = \{(v_5, v_4), (v_1, v_3)\} = \{v_5, v_4, v_1, v_3\}.$

$\chi_4 = \{e_7\} = \{(v_2, v_4)\} = \{v_2, v_4\}.$

From Remark 2.7. Any vertex of degree one can't be included in standard of partition. If we consider $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9$ is standard of partition, then we have $w = 9$, and $t_1 = 8$, $t_2 = 9$, $t_3 = 4$, and $t_4 = 2$, but $t_3 + t_4 = 6 < w = 9$, this contradicts Proposition 2.8. Therefore we consider v_1, v_2, v_3, v_4, v_5 is standard of partition, then we have $w = 5$, and $t_1 = 4$, $t_2 = 4$, $t_3 = 4$, $t_4 = 2$, and $w < t_i + t_j$, where $1 \le i < j \le 4$.

Of $\Gamma_1, \Gamma_2, \Gamma_3$ consists of two edges, the class Γ_4 consists of only one edge, this common edge.

IV. Partitioning of Graph and Subgraph into Colour Classes

In this section we introduce some results related to method of finding minimum number of colour classes for edge coloring, see Proposition 4.1. [15]. we also introduce concept of quasi-non common edge, and introduce some results related to this concept.

Remark 4.1. Families of disjoint sets colouring technique using concept of minimum number of colour classes and concept of maximum number of colour classes, if it is mentioned partitioned into colour classes it means partitioned into minimum colour classes.

Proposition 4.2. Suppose G_1 is a subgraph of a graph G_2, G_1 be a graph of k edges $e_1, e_2, e_3,..., e_k$, and G_2 be a graph of l edges $e_1, e_2, e_3,..., e_k, e_{k+1}, e_{k+2},..., e_l$, where $k < l$, each of G_1 and G_1^c with maximum degree of vertex equal n, where $G_1 \cup G_1^c = G_2$. If these k edges partitioned into m_1 minimum colour classes and the edges $e_1, e_2, e_3,..., e_k, e_{k+1}, e_{k+2},..., e_l$ partitioned into m minimum colour classes, then $n \le m_1 \le m$.

Proof: Let G_2 be a graph of l edges $e_1, e_2, e_3,..., e_k, e_{k+1}, e_{k+2},..., e_l$, and these k edges $e_1, e_2, e_3,..., e_k$ partitioned into m minimum colour classes $\chi_1, \chi_2, \chi_3,..., \chi_m$. Any one of the edges $e_{k+1}, e_{k+2},..., e_l$ either belongs to one of classes $\chi_1, \chi_2, \chi_3,..., \chi_{m_1}$, or does not belong to any one of classes $\chi_1, \chi_2, \chi_3,..., \chi_{m_1}$. Therefore either these l edges partitioned into m_1 minimum colour classes or these l edges partitioned into minimum number of colour classes greater than m_1, then $m_1 \le m$, since G_1 with maximum degree of vertex equal n, and since G_1^c with maximum degree of vertex equal n, then we have $n \le m_1$ and $n \le m$. Therefore $n \le m_1 \le m$.

Proposition 4.3. Suppose G_1 is a subgraph of a graph G_2, G_1 be a graph of k edges $e_1, e_2, e_3,..., e_k$, and G_2 be a graph of l edges $e_1, e_2, e_3,..., e_k, e_{k+1}, e_{k+2},..., e_l$, where $k < l$, G_1 each

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of G_1 and G_1^c with maximum degree of vertex equal n, where $G_1 \cup G_1^c = G_2$. If these l edges partitioned into m_2 minimum colour classes, and the edges $e_1, e_2, e_3, ..., e_k$ partitioned into m minimum colour classes, then $n \le m \le m_2$.

Proof: Let G_1 be a graph of k edges $e_1, e_2, e_3, ..., e_k$, and G_2 be a graph of l edges $e_1, e_2, e_3, ..., e_k, e_{k+1}, e_{k+2}, ..., e_l$, suppose these l edges partitioned into m_2 minimum colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_{m_2}$. If these k edges $e_1, e_2, e_3, ..., e_k$ partitioned into m minimum colour classes, then either $e_1, e_2, e_3, ..., e_k$ belongs to the classes $\chi_1, \chi_2, \chi_3, ..., \chi_{m_2}$, or belong to some of classes $\chi_1, \chi_2, \chi_3, ..., \chi_{m_2}$, then we have $m \le m_2$, since G_1 with maximum degree of vertex equal n, then we have $n \le m$, and since G_2 with maximum degree of vertex equal n, then we have $n \le m_2$. Therefore $n \le m \le m_2$.

Definition 4.4. Let G be a graph of edges $e_1, e_2, e_3, ..., e_k$ and $v_1, v_2, v_3, ..., v_w$ are proper vertices of these k edges, and $G = G_1 \cup G_2$, such that G_1 consists of k_1 edges and w_1 proper vertices, and G_2 consists of k_2 edges and w_2 proper vertices, where $k_1 + k_2 = k$, an edge e_l incident between v_i and v_j, where $v_j \in (G_1 \cap G_2), 1 \le l \le k_1, 1 \le i < j \le w$, after disconnection of G_1 and G_2, we have $v_j \in G_1$, and $a(e_l) = 1$, where $a(e_l)$ is appearance of the edge e_l at w_1 proper vertices, then the edge e_l is called quasi non common edge w.r.t. G_1.

We introduce the following proposition without proof, since proof is direct from definition of quasi non common edge and using Proposition 2.8.

Proposition 4.5. Let $G = G_1 \cup G_2$, and e_l be quasi non common edge w.r.t. G_1. after disconnection of G_1 and G_2, then e_l will be non-common edge in G_1.

Proposition 4.6. Let G be a graph of edges $e_1, e_2, e_3, ..., e_k$ and $v_1, v_2, v_3, ..., v_w$ are proper vertices of these k edges, and $G = G_1 \cup G_2$, such that G_1 consists of k_1 edges and

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$v_1, v_2, v_3, ..., v_t$ are proper vertices of these k_1 edges, with maximum degree of vertex equal $n,$ if the edge e_l is quasi non common edge w.r.t. $G_1,$ $1 \le l \le k_1,$ and $v_1, v_2, v_3, ..., v_t$ is standard of partition these k_1 edges into m minimum colour classes, where $n+1 \le m,$ if we remove e_l from $G_1,$ then $G_1 - e_l$ partition into m minimum colour classes.

Proof: Let G_1 consists of k_1 edges partitioned into m minimum colour classes. Let e_l be quasi non common edge w.r.t. $G_1,$ from definition of quasi non common edge G_1 is a graph, with e_l is non common edge, if we using Proposition 2.8. Then $G_1 - e_l$ partitioned into m minimum colour classes.

V. Some Results Related to Maximum Degree of Vertex

In this section we introduce some results explain some properties of standard of partitioning edges into colour classes, properties related to maximum degree of vertex.

Proposition 5.1. Let G be a graph of k common edges and $v_1, v_2, v_3, ..., v_w$ are proper vertices of these k edges, with maximum degree of vertex equal $n,$ where $2 \le n,$ if G is simple graph then $n < w,$ and if is multiple graph then $w \le n$ and $n < w.$

Proof: Let G be simple complete graph, from definition of complete graph $n < w.$ If we remove edges of G such that number of vertices remain fixed $w,$ and maximum degree of vertex remains fixed $n,$ such that the remaining edges all are common, and w vertices are proper vertices of these remaining edges, then G simple and $n < w.$ Let G be a complete graph, then $n < w,$ if we add edges to G such that number of vertices remain fixed $w,$ then G become multiple graph and $w \le n.$ Let G be a graph of k edges $e_1, e_2, e_3, ..., e_k$ and $v_1, v_2, v_3, ..., v_w$ are proper vertices of these k edges, where $n+2 < w.$ If we add an edge e_{k+1} incident between v_1 and $v_2,$ where the edge e_1 also incident between

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v_1 and v_2, then G become multiple graph where $n+1<w$, therefore G is multiple graph and $n<w$.

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Proposition 5.2. Let G be a simple graph of k common edges and $v_1, v_2, v_3, ..., v_w$ are standard of partition of these k edges, with maximum degree of vertex equal n, where w is odd, and n is even, if these k edges partitioned into $n+1$ minimum colour classes, then there exist at least three vertices each of degree n, and remaining vertices each of degree $n-1$, with condition $w \le n+1$.

Proof: Suppose there is only one vertex of degree n, then we have $\sum_{j=1}^{w} d(v_j) = n+(n-1)(w-1)$, using Proposition2.6. And definition of standard of partition, we have $(w-1)n+2 \le (w-1)n+2x = n+(n-1)(w-1)$, then we have $w+1 \le n$, this contradicts Proposition 5.1. Suppose there are two vertex of degree n, then we have $(w-1)n+2 \le (w-1)n+2x = 2n+(n-1)(w-2)$, then we have $w \le n$, this contradicts Proposition 5.1. Therefore the result not holds only one vertex of degree n, and not holds for two vertices of degree n. Suppose there are three vertices of degree n, then $(w-1)n+2 \le (w-1)n+2x = 3n+(n-1)(w-3)$, then we have $w \le n+1$. Therefore the result not holds.

Proposition 5.3. Let G be a simple graph of k common edges and $v_1, v_2, v_3, ..., v_w$ are standard of partition of these k edges, with maximum degree of vertex equal n, where w is odd, and n is odd, if these k edges partition into $n+1$ minimum colour classes, then there exist at least four vertices each of degree n, and remaining vertices each of degree $n-1$, with condition $w \le n+2$.

Proof t: Suppose there is only one vertex of degree n, then we have $\sum_{j=1}^{w} d(v_j) = n + (n-1)(w-1)$, using Proposition 2.6. And definition of standard of partition, we have $(w-1)n + 2 \leq (w-1)n + 2x = n + (n-1)(w-1)$, then we have $w+1 \leq n$, this contradicts Proposition 5.1. Suppose there are two vertex of degree n, then we have $(w-1)n + 2 \leq (w-1)n + 2x = 2n + (n-1)(w-2)$, then we have $w \leq n$, this contradicts Proposition 5.1. Therefore the result not holds only one vertex of degree n, and not holds for two vertices of degree n. Suppose there are three vertices of degree n, then $(w-1)n + 2 \leq (w-1)n + 2x = 3n + (n-1)(w-3)$, then we have $w \leq n+1$, this contradicts Proposition 5.1. and since both where w and n are odd, we have to show $w \leq n+2$. Therefore the result not holds for three vertex of degree n. Suppose there are four vertices of degree n, then we have $(w-1)n + 2 \leq (w-1)n + 2x = 4n + (n-1)(w-4)$, then we have $w \leq n+2$. Therefore the result holds.

The Following Two Examples Satisfying Proposition 5.3.

Example 5.4. Let G be a graph of 16 common edges, and 7 vertices, four vertices each of degree 5, and three vertices each of degree 4. These 16 edges partitioned into 6 minimum colour classes.

Example 5.5. Let G be a graph of 15 common edges, and 7 vertices, three vertices each of degree 5, three vertices each of degree 4, and one vertex of degree 3. These 15 edges partitioned into 5 minimum colour classes.

The following two examples satisfying Proposition 5.2.

Example 5. 6. Let G be a graph of 19 common edges, and 7 vertices, three vertices each of degree 6, and four vertices each of degree 5. These 19 edges partitioned into 7 minimum colour classes.

Example 5.7. Let G be a graph of 18 common edges, and 7 vertices, two vertices each of degree $6,$ four vertices each of degree $5,$ and one vertex of degree $4.$ These 18 edges partitioned into 6 minimum colour classes.

The following proposition is generalization of proposition 5.2 and Proposition 5.3. The proof is the same as in proposition 5.2 and Proposition 5.3.

Proposition 5.8. Let G be a simple graph of k common edges and $v_1, v_2, v_3, \dots, v_w$ are standard of partition of these k edges, with maximum degree of vertex equal $n,$ where w is odd, if these k edges partitioned into $n+1$ minimum colour classes, then there exist at least x vertices each of degree $n,$ and remaining vertices each of degree $n-1,$ with condition $w \le n+(x-2).$

VI. Some Results Related to Maximum Number of Edges Belgon to Colour Class

In this section we introduce some results about relation between maximum number of edges belongs to each colour class, and standard of partitioning edges into minimum number of colour classes.

Example 6.1. Let G be a graph of k common edges and $v_1, v_2, v_3, \dots, v_w$ are proper vertices of these k edges, with maximum degree of vertex equal $n,$ where w is odd, n is even, and $v_1, v_2, v_3, \dots, v_w$ is standard of partitioning these k edges into $n+1$ minimum colour classes, if $\sum_{j=1}^{w} d(v_j) = n(w-1)+2x,$ where $2x \le n.$ We want to delete all edges of the graph $G,$ step by step, such that in each step we delete $\frac{1}{2}(w-1)$ edges, and each one of $w-1$ vertices of maximum degree reduced by one. Since n is even, using Proposition 4.2. then there exist at least three vertices each of degree $n.$ In the following table we explain each step. Suppose these k common edges partitioned into

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m minimum colour classes $\chi_1,\chi_2,\chi_3,...,\chi_m$. the table below satisfies Theorem2.9. If number of vertices is odd and degree of vertex is odd.

	Before Removal of a colour Class χ_i	**After Removal of a colour Class χ_i**
Removal of χ_1	3 vertices each of degree n	$(w-1-3)$ vertices each of degree $(n-1)$
Removal of χ_2	4 vertices each of degree $n-1$	$(w-1-4)$ vertices each of degree $(n-2)$
Removal of χ_3	5 vertices each of degree $n-2$	$(w-1-5)$ vertices each of degree $(n-3)$
Removal of χ_i	$i+3$ vertices each of degree $n+1-i$	$(w-1-i-3)$ vertices each of degree $(n-i)$
Removal of χ_n	$n+3$ vertices each of degree $n+1-n$	$(w-1-n-3)$ vertices each of degree zero
Removal of χ_{n+1}	$n-1+3$ vertices each of degree zero	$(w-1-3)$ vertices each of degree $(n-1)$

From definition of common edges and proper vertices, directly can introduce the following remark.

Remark 6.2. Let G be a simple graph of k common edges and w vertices are proper vertices of these k edges, and w is odd. If we remove $\frac{1}{2}(w-1)$ edges, each two of them are not adjacent, then $w-1$ vertices its degree reduced by one.

Proposition 6.3. Let G be a simple graph of k common edges and $v_1,v_2,v_3,...,v_w$ are proper vertices of these k edges, with maximum degree of vertex equal n, where w is odd, if we remove edges step by step, using $n+1$ steps, such that at each step $w-1$ vertices its degree reduced by one, and at each step we remove $\frac{1}{2}(w-1)$ edges, before final removal every vertex either of degree one or degree zero,(at least one vertex of degree zero). Then these k edges partitioned into $n+1$ minimum colour classes, and $v_1,v_2,v_3,...,v_w$ are standard of partition these k edges into minimum colour classes.

Proof: From Remark 6.2. Removing of $\frac{1}{2}(w-1)$ edges reduced degree of each one of these $w-1$ vertices by one, then every two of these $\frac{1}{2}(w-1)$ edges are not adjacent, then all these $\frac{1}{2}(w-1)$ edges can form one colour class. At final step suppose remove x edges, and these x edges can form one colour class. This satisfies conditions of Proposition 4.1. [15] Then these k edges partitioned into $n+1$ minimum colour classes. Since $t_1 = t_2 = t_3 = ... = t_n = w-1,$ and $t_{n+1} = 2x,$ where $2x \le w-1,$ and since $v_1, v_2, v_3, ..., v_w$ satisfies $w+1 \le t_i + t_j,$ where $1 \le i < j \le n+1,$ then $v_1, v_2, v_3, ..., v_w$ are standard of partition these k edges into minimum colour classes.

Proposition 6.4 Let G be a graph of k common edges and $v_1, v_2, v_3, ..., v_w$ are proper vertices of these k edges, with maximum degree of vertex equal $n,$ where w is odd, and $v_1, v_2, v_3, ..., v_w$ are standard of partition these k edges, if $\sum_{j=1}^{w} d(v_j) = n(w-1)+2,$ and if these k edges partitioned into $n+1$ minimum colour classes, then $t_1 = t_2 = t_3 = ... = t_n = w-1,$ and $t_{n+1} = 2.$

Proof: Suppose these $n+1$ colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_{n+1},$ labelled by $1,2,3,..., n+1$ respectively. Suppose $d(v_1) = d(v_2) = d(v_3) = ... = d(v_{w-1}) = n,$ $d(v_w) = 2,$ and $v_i \equiv (1,2,3,..., n),$ $v_1 \equiv (1,2,3,..., n-1, n+1),$ $v_w \equiv (n, n+1),$ where $2 \le i \le w-1,$ then we have $t_1 = t_2 = t_3 = ... = t_n = w-1,$ and $t_{n+1} = 2,$ this means $v_1, v_2, v_3, ..., v_w$ are standard of partition and there are $n+1$ minimum colour classes.

Suppose $d(v_1) = d(v_2) = d(v_3) = ... = d(v_{w-1}) = n,$ $d(v_w) = 2,$ and $v_j \equiv (1,2,3,..., n),$ $v_1 \equiv (1,2,3,..., n-1, n+1),$ $v_w \equiv (n, n+1),$ where $2 \le j \le w-1,$ before changing degree of all vertices. Suppose there is an edge incident between v_r and v_s belongs to the colour class χ_i, where $1 \le i \le n+1$, we change this edge to be incident between v_r and v_w such that all edges at v_w not adjacent to this edge. The following condition satisfy changes of degree for all vertices: $d(v_j) = n - p_j$, $d(v_w) = 2 + p_1 + p_2 + p_3 + ... + p_{w-1}$, where $1 \le j \le w-1$, and $0 \le p_j \le n-2$. Since degree of vertices changes but labels of any colour class remain fixed, then we have $t_1 = t_2 = t_3 = ... = t_n = w-1$, and $t_{n+1} = 2$.

We introduce the following proposition without proof, it is a generalization of Proposition 6.4.

Proposition 6.5. Let G be a graph of k common edges and $v_1, v_2, v_3, ..., v_w$ are proper vertices of these k edges, with maximum degree of vertex equal n, where w is odd, and $v_1, v_2, v_3, ..., v_w$ are standard of partition these k edges, if $\sum_{j=1}^{w} d(v_j) = n(w-1) + 2x$, where $2x \le n$, and if these k edges partitioned into $n+1$ minimum colour classes, then

$t_1 = t_2 = t_3 = ... = t_n = w-1$, and $t_{n+1} = 2x$.

VII. Some Results Related to Standard of Partition

In this section we study cases of relations between standard partition of graph and standard partition of subgraphs.

Remark 7.1. Let G be a graph of edges $e_1, e_2, e_3, ..., e_k$ and $v_1, v_2, v_3, ..., v_w$ are proper vertices of these k edges, if these k edges partition into minimum m colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, if there exist two colour classes χ_i and χ_j such that $t_i + t_j \le w$, where $1 \le i < j \le m$, then from definition of standard of partition $v_1, v_2, v_3, ..., v_w$ is not of standard of partition for these k edges.

Proposition 7.2. Let G be a graph of edges $e_1, e_2, e_3, ..., e_k$ and $v_1, v_2, v_3, ..., v_w$ are proper vertices of these k edges, if these k edges partitioned into minimum m colour classes

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$\chi_1, \chi_2, \chi_3,..., \chi_m$, then either $v_1, v_2, v_3,..., v_w$ is standard of partition for these k edges, or there exists a proper subset of $\{v_1, v_2, v_3,..., v_w\}$ as standard of partition for a proper subset of $\{e_1, e_2, e_3,..., e_k\}$.

Proof: First case supposes these k edges partitioned into minimum m colour classes $\chi_1, \chi_2, \chi_3,..., \chi_m$, and there is no existence of standard of partition for these k edges. From definition of standard of partition for these k edges, then there exist two colour classes χ_i and χ_j such that $t_i + t_j \leq w$, then the two colour classes χ_i and χ_j can form one colour class, then these k edges partitioned into $m-1$ colour classes, this contradicts m is minimum number of colour classes. In second case suppose for any G^* where $G^* \subset G$, and G^* is graph of w^* vertices and k^* edges, where these k^* edges of G^* is proper subset of $\{e_1, e_2, e_3,..., e_k\}$. Suppose these k^* edges partition into minimum m colour classes $\chi_1, \chi_2, \chi_3,..., \chi_m$, and there is no existence of standard of partition for these k^* edges. From definition of standard of partition for these k^* edges, then there exist two colour classes χ_i and χ_j such that $t_i + t_j \leq w^*$, then these k^* edges partition into $m-1$ colour classes, this contradicts m is minimum number of colour classes. Therefore, the result holds.

No need to prove the following corollary, it is direct application of Proposition 7.2.

Corollary 7.3. Let G be a graph of edges $e_1, e_2, e_3,..., e_k$ and $v_1, v_2, v_3,..., v_w$ are proper vertices of these k edges, if these k edges partitioned into minimum m classes $\chi_1, \chi_2, \chi_3,..., \chi_m$, then either $w+1 \leq t_i + t_j$ or $w^* + 1 \leq t_i^* + t_j^*$, for all i, j, where $1 \leq i < j \leq m$, and $u_1, u_2, u_3,..., u_{w^*}$ are proper vertices of edges $o_1, o_2, o_3,..., o_{k^*}$, such that $\{u_1, u_2, u_3,..., u_{w^*}\} \subset \{v_1, v_2, v_3,..., v_w\}$, $\{o_1, o_2, o_3,..., o_{k^*}\} \subset \{e_1, e_2, e_3,..., e_k\}$, and $k^* < k$, $w^* < w$.

Remark 7.4. Let G be a graph of edges $e_1, e_2, e_3,..., e_k$ and $v_1, v_2, v_3,..., v_w$ are proper vertices of these k edges, if $v_1, v_2, v_3,..., v_w$ is standard of partition the edges $e_1, e_2, e_3,..., e_k$ into minimum m colour classes, usually we say G is standard of partitioning its edges into minimum m colour classes. Also if $G^* \subset G,$ and G^* is a graph of k^* edges and w^* vertices are proper vertices of these k^* edges, usually we say G^* is standard of partitioning its edges into minimum m colour classes.

Proposition 7.5. Let G be a graph of edges $e_1, e_2, e_3, ..., e_k$ and $v_1, v_2, v_3, ..., v_w$ are proper vertices of these k edges, if $G^* \subset G,$ and G^* is a graph of w^* vertices and k^* edges. G^* is only subgraph of G such that G^* is standard partitioning its edges into minimum m colour classes, and w^* is standard of partition for these k^* edges. If G and any subgraph of $G,$ except $G^*,$ is not standard of partitioning its edges into minimum m or $m+p,\ 1 \le p$ colour classes, then these k edges partitioned into minimum m colour classes.

Proof: Using Proposition 4.3. Then G can not partitioned into minimum $m-p,\ 1 \le p$ colour classes. Using Proposition 4.2. Then G partitioned into minimum m or $m+p,\ 1 \le p$ colour classes. Using Proposition 7.2. Then G partitioned into minimum $m.$

Proposition 7. 6. Suppose G_1 is a subgraph of a graph G_2, G_1 be a graph of k common edges $e_1, e_2, e_3, ..., e_k,$ and $v_1, v_2, v_3, ..., v_{w_1}$ are proper vertices of these k edges. G_2 be a graph of l edges $e_1, e_2, e_3, ..., e_l,$ and $v_1, v_2, v_3, ..., v_{w_2}$ are proper vertices of these l edges, where $k < l.$ If each one of $v_1, v_2, v_3, ..., v_{w_1}$ is of degree less than $n,$ then $v_1, v_2, v_3, ..., v_{w_1}$ is not standard of partition for edges in $G_2.$

Proof: Let each one of $v_1, v_2, v_3, ..., v_{w_1}$ is of degree $n-1,$ using Proposition 6.4. Then these k common edges $e_1, e_2, e_3, ..., e_k$ partitioned into n colour classes, or $n-1$ colour classes. If the edges $e_1, e_2, e_3, ..., e_l$ partitioned into n colour classes, then standard of partition for these l edges will be a one vertex in $G_2,$ with degree equal $n.$ If the edges $e_1, e_2, e_3, ..., e_l$

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partitioned into $n+1$ colour classes, then standard of partition for these l edges will be vertices in G_2, and not in G_1. Then $v_1, v_2, v_3, \ldots, v_{w_1}$ is not standard of partition for edges in G_2.

Proposition 7.7. Let G be a graph of edges $e_1, e_2, e_3, \ldots, e_k$ and $v_1, v_2, v_3, \ldots, v_w$ are proper vertices of these k edges, with maximum degree of vertex equal n, and w is odd, if these k edges partitioned into $n+1$ colour classes, and $G = G_1 \cup G_2$, such that G_1 consists of k_1 common edges and w_1 vertices be proper vertices of these k_1 edges, edges of G_1 partitioned into minimum $n+1$ colour classes,, and G_2 consists of k_2 common edges and w_2 vertices be proper vertices of these k_2 edges, and $w_1 \le w_2$, edges of G_2 partitioned into minimum $n-1$ colour classes, then w vertices of G can't be standard of partition for k edges of G.

Proof: Given w_1 vertices of G_1 be standard of partition of G_1, suppose edges of G_1 be partitioned into $n+1$

colour classes $\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_{n+1}$. Suppose edges of G_2 be partition into $n-1$ classes $\Psi_1, \Psi_2, \Psi_3, \ldots, \Psi_{n-1}$. Suppose edges of G be partitioned into $n+1$ colour classes $\chi_1, \chi_2, \chi_3, \ldots, \chi_{n+1}$, such that $\chi_1 = \Gamma_1 \cup \Psi_1$, $\chi_2 = \Gamma_2 \cup \Psi_2$, $\chi_3 = \Gamma_3 \cup \Psi_3$, $\chi_{n-1} = \Gamma_{n-1} \cup \Psi_{n-1}$, $\chi_n = \Gamma_n$, and $\chi_{n+1} = \Gamma_{n+1}$. Suppose each colour class has maximum t_i, where $1 \le i \le n+1$, then we have $t_1 = t_2 = t_3 = \ldots = t_{n-1} = w-1$, and $t_n = t_{n+1} = w_1 - 1$, since $w_1 \le w_2$, then there exist χ_i and χ_j, such that $t_n + t_{n+1} \le \frac{w-1}{2} + \frac{w-1}{2} = w-1$, therefore w vertices of G can't be standard of partition for edges of G.

Proposition 7.8. Let G be a graph of edges $e_1, e_2, e_3, \ldots, e_k$ and $v_1, v_2, v_3, \ldots, v_w$ are proper vertices of these k edges, where w_1 is odd, with maximum degree of vertex equal n, if these k edges partitioned into $n+1$ colour classes, and $G = G_1 \cup G_2$, such that G_1 consists of k_1 common edges and w_1 vertices be proper vertices of these k_1 edges, edges of G_1 partitioned into $n+1$ colour classes, if edges of G_2 partition into n colour classes, and $w_1 \leq \frac{w-1}{2} \geq n,$ then w vertices of G can't be standard of partition for edges of $G.$

Proof: Given w_1 vertices of G_1 be standard of partition of G_1, suppose edges of G_1 be partitioned into $n+1$

colour classes $\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_{n+1}$. Suppose edges of G_2 be partitioned into n classes $\Psi_1, \Psi_2, \Psi_3, ..., \Psi_n$. Suppose edges of G be partition into $n+1$ colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_{n+1}$, such that $\chi_1 = \Gamma_1 \cup \Psi_1,$ $\chi_2 = \Gamma_2 \cup \Psi_2,$ $\chi_3 = \Gamma_3 \cup \Psi_3,$ $\chi_n = \Gamma_n \cup \Psi_n,$ an $\chi_{n+1} = \Gamma_{n+1}$. From $\sum_{j=1}^{w} d(v_j) \leq wn = n(w-1)+n,$ then there are two colour classes χ_n and χ_{n+1}, such that $t_n + t_{n+1} \leq \frac{w-1}{2} + \frac{w-1}{2} = w-1,$ therefore w vertices of G can't be standard of partition for edges of $G.$

Conclusion

Standard of partition edges into minimum number of colour classes have the following properties:

1. For any two colour classes χ_i and χ_j, we have $w+1 \leq t_i + t_j$.
2. If w standard of partition edges, and w is odd, and n is odd, if these edges partitioned into $n+1$ minimum colour classes, then there exist at least four vertices each of degree $n.$

   If w standard of partition edges, and w is odd, and n is odd, if these edges partitioned into $n+1$ minimum colour classes, then there exist at least four vertices each of degree $n.$

3. Let G be a graph of edges $e_1, e_2, e_3, ..., e_k$ and $v_1, v_2, v_3, ..., v_w$ are proper vertices of these k edges, if these k edges partitioned into minimum m colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, then either $v_1, v_2, v_3, ..., v_w$ is standard of partition for these k edges, or there exists a subset of $\{v_1, v_2, v_3, ..., v_w\}$ as standard of partition for a proper subset of $\{e_1, e_2, e_3, ..., e_k\}$.
4. For a graph G and subgraph G^*, either $w+1 \le t_i + t_j$ or $w^* + 1 \le t_i + t_j$, for all i, j, where $1 \le i < j \le m$.

References

1. Bondy and Murty, "Graph Theory with applications". Elsevier Science Publishing Co., Inc.1982.
2. Reinhard Diestel, "Graph Theory". Electronic 2000, Spring-Verlag New York 1997, 2000.
3. Frank Harary, "Graph Theory". Addison-Wesley Publication Company, Inc. 1969.
4. Hassan, M.E., "Family of Disjoint Sets and its Applications". IJIRSET Vol. 7. Issue 1, Jan 2018, 362-393.
5. Hassan, M.E., Colouring of Graphs Using Colouring of Families of Disjoint Sets Technique". IJIRSET Vol. 7, Issue 10, October 2018, 10219-10229. Colouring.
6. Hassan, M.E., "Types of Colouring and Types of Families of Disjoint Sets". IJIRSET Vol. 7. Issue 11, Nov 2018, 362-393.
7. Hassan, M.E., "Colouring of finite Sets and Colouring of Edges finite Graphs". IJIRSET Vol. 7. Issue 12, Dec 2018, 11663-11675.
8. Hassan, M.E., "Sorts of Colour Classes and Sorts of Families of Disjoint Sets". IJIRSET Vol. 8. Issue 1, Jan2019, 56-63.
9. Hassan, M.E., "Trivial Colouring and Non-Trivial Colouring for Graph's Edges". IJIRSET Vol. 8. Issue 2, Feb2019, 1014-1024.
10. Hassan, M.E., "Some Results of Edge Colouring Using Family of Disjoint Colouring Technique". IJIRSET Vol. 8. Issue 4, April2019, 4667-4675.

11. Hassan, M.E. “Trivial Colouring and Non-Trivial Colouring for Graph’s Vertices”. IJIRSET Vol. 8. Issue 6, June2019, 7398-7410.
12. Hassan, M.E., “Families of Disjoint Sets Colouring Technique and Concept of Common Set and Non-Common Set”. IJIRSET Vol. 10. Issue 7, July10491- 10507.
13. Hassan, M.E., “Families of Disjoint Sets Colouring Technique and Concept of Common and Non-Common edge “IJIRSET Vol.10 10. Issue 9, September2021. (13280-13296).
14. Hassan, M.E., “Families of Disjoint Sets Colouring Technique and Concept of Common and Non-Common Vertex” IJFMR23068622. Volume 5, Issue 6 November-December 2023. (1-14).
15. Hassan, M.E., “How to Maximize Minimum Number of Colour Classes For Edge Colouring Using Families of Disjoint Sets Colouring Technique Sets", IJSR, Volume 3, No 12 Dec 2024. (214-230).
16. Hassan, M.E., “Families of Disjoint Sets Colouring Technique and Concept of Common Face and Non-Common Face", IJSR, Volume 4, No1 Jan 2025. (323-345).
17. Oystein Ore, “The Four-Color Problem". ACADEMIC PRESS, New York, London, 1967.
18. Douglas B. West, “Introduction to Graph Theory". Second Edition, Department of Mathematics Illinois University, (2001).
19. Wilson, R., “Introduction to Graph Theory". Fourth Edition, Addison Wesley Longman Limited, England, (1996).