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Families of Disjoint Sets Colouring Technique and Concept of Common Face and Non-Common Face

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Abstract

Families of disjoint sets colouring technique is trial to generalize all type of colouring and partition. This paper related to face colouring, in this paper we introduce the concept of common face and non-common face, and used two methods to determine adjacency of faces. We introduce some results related to the concept of common face and results related to non-common face, and introduce some results explain the number of minimum colour classes not changed after addition or after removal of non-common face, if maximum clique number is constant.

Keywords: Families of Disjoint Sets Colouring Technique, Common Face, Proper Clique, Improper Clique.

1. Introduction

In paper [4], we introduced the concept of families of disjoint sets colouring technique, and published papers [5], [6], [7], [8], [9], [10], [11] to introduce some results related to this colouring (partition) technique.

In paper [12], we introduce the concept of common set to establish some results for families of disjoint sets colouring technique for partition sets. The concept of common set is essential concept for families of disjoint sets colouring technique, and this was reason for modifications of some definitions and some results in previous papers.

Paper [13] analogue to paper [12]. In paper [13], we introduce the concept of common edge between two vertices, as essential concept for edge colouring to establish some results of edge colouring.

In paper [14], we introduce concept of common vertex and quasi non common vertex, and we introduce some results related to these concepts. Also, in Paper [14] we introduced a result for method used to partition vertices into minimum number of colour classes.

In Paper [15], we prove a result for method of finding minimum number of colour classes for edge colouring, and explain how to find different values of minimum colour classes, for edge colouring, when maximum degree of vertex is constant, and introduce a condition to maximize minimum number of colour classes, and use multiple edges as an example.

In this paper (section three), we introduce the concept of common face and non-common face and proper clique (related to faces). We introduce some results related to the concept of the common face and results related to non-common face, and introduce some results explain the number of minimum colour classes not changed after addition or after removal of non-common face, if maximum clique number is constant. In section four, we introduce a method of finding the minimum number of colour classes for face colouring. In section five, we introduce the concept of common face, non-common face related to proper vertex separates faces. In section six we introduce some definitions and results about multiple vertices and multiple faces.

2. Related Work

In this section we write some definitions and results be needed through this paper.

- Theorem 2.1. The sum of degrees of vertices equals twice the number of edges [17].
- Definition 2.2.A clique of a graph is a maximal complete sub graph. A clique number of a graph G is largest number n such that K_n is a subgraph of G [3].
- Definition 2.3. degree of a vertex v in graph G is the number of edges of G incident
- With v, each loop counting as two edges [3].
- Definition 2.4. We will refer to regions defined by a plane graph as its face, the unbounded region being called the exterior face [3].
- Remark 2.5. Each edge is on the boundary of two neighboring faces. If no two faces that are separated by an edge -have the same colour [1].
- Definition 2.6. Two graphs are isomorphic if there exists a one-to-one correspondence between their points sets which preserves adjacency [3].
- Definition 2.7. The valence of a face F is the number of boundary edges for F [16].

3. Common Face and Non Common Face Related to Clique:

In this section we introduce the concept of common face, non-common face, proper clique (related to faces) and introduce some results related to these two concepts. In this section two adjacent faces means they have common edge).

- Definition 3.1. Let G be a graph of k faces, C_j is called clique of faces $f_1, f_2, f_3,..., f_n$ if any two faces are adjacent, and any face f_i not adjacent to all faces

$f_1, f_2, f_3, ..., f_n,$ and the face f_i can be adjacent to some faces of $f_1, f_2, f_3, ..., f_n,$ where $n+1 \le i \le k,$ and C_j is called with clique number $n,$ or C_j is clique with degree $n.$

- Remark 3.2. If C_j is clique of faces $f_1, f_2, f_3, ..., f_n,$ then we used the notation $C_j = \{f_1, f_2, f_3, ..., f_n\}$ or $C_j \equiv (f_1, f_2, f_3, ..., f_n).$
- Definition 3.3. Let C_1 be clique of the faces $\{f_1, f_2, f_3, ..., f_t\},$ then C_1 is called improper clique of faces $\{f_1, f_2, f_3, ..., f_t\},$ if there exists a clique C_2 of the faces $\{f_1^*, f_2^*, f_3^*, ..., f_s^*\}$ such that $\{f_1, f_2, f_3, ..., f_t\} \subset \{f_1^*, f_2^*, f_3^*, ..., f_s^*\},$ $t < s,$ and $C_1 \subset C_2,$ and for all i and some j we have $f_i = f_j^*,$ where $C_1 = \{f_1, f_2, f_3, ..., f_t\},$ $C_2 = \{f_1^*, f_2^*, f_3^*, ..., f_s^*\}, 1 \le i \le t$ and $1 \le j \le s.$
- Definition 3.4. Let C_1 be clique of the faces $\{f_1, f_2, f_3, ..., f_t\},$ then C_1 is called proper clique of faces $\{f_1, f_2, f_3, ..., f_t\},$ if there is no a clique C_2 of the faces $\{f_1^*, f_2^*, f_3^*, ..., f_s^*\}$ such that $\{f_1, f_2, f_3, ..., f_t\} \subset \{f_1^*, f_2^*, f_3^*, ..., f_s^*\},$ $t < s,$ and $C_1 \subset C_2,$ and for all i and some j we have $f_i = f_j^*,$ where $C_1 = \{f_1, f_2, f_3, ..., f_t\},$ $C_2 = \{f_1^*, f_2^*, f_3^*, ..., f_s^*\}, 1 \le i \le t$ and $1 \le j \le s.$
- Definition 3.5. In a graph G if each of C_1 and C_2 be proper clique of the face $f,$ then f is called common between two cliques C_1, C_2 and $C_1 \ne C_2.$ A face f is called common face between t proper cliques $C_1, C_2, C_3, ..., C_t$ if $C_i \ne C_j$, for all i, j where $1 \le i < j \le t$ and each one of $C_1, C_2, C_3, ..., C_t$ is proper clique of $f.$
- Definition 3.6. In a graph G the face f is called non common between any number of cliques, if there is only one clique include all faces adjacent to f or f is isolated face. This means a non-common face belongs to only one clique.
- Remark 3.7. In a graph G the face f is called non common between any number of cliques, if f is isolated face.

- Example 3.8. Let G be a graph of nine faces $f_1, f_2, f_3, ..., f_9$ and three clique C_1, C_2, C_3, each clique with clique number equal four. C_1 is clique of four faces f_1, f_2, f_3, f_4, , C_2 is clique of four faces define f_4, f_5, f_6, f_7 if define C_3 is clique of four faces f_7, f_8, f_9, f_1. We use for cliques notations $C_1 = (f_1, f_2, f_3, f_4), C_2 = (f_4, f_5, f_6, f_7)$, $C_3 = (f_7, f_8, f_9, f_1)$. The face f_4 is common face between the two proper cliques C_1, C_2 , the face f_7 is common face between the two proper cliques C_2, C_3 and f_1 is common face between the two proper cliques C_3, C_1. The faces $f_2, f_3, f_5, f_6, f_8, f_9$ are non-common faces. If these nine faces partition into four colour classes $\chi_1, \chi_2, \chi_3, \chi_4$ where $\chi_1 = \{f_1, f_5\}$, $\chi_2 = \{f_4, f_8\}$, $\chi_3 = \{f_2, f_7\}$, and $\chi_4 = \{f_3, f_6, f_9\}$.
- Definition 3.9. Let C_1 be clique of the faces $f_1, f_2, f_3, ..., f_t$, then C_1 is called isolate clique of faces $f_1, f_2, f_3, ..., f_t$ if each one of $f_1, f_2, f_3, ..., f_t$ is non common face.
- Definition3.10. Let G be a graph of k common and non-common faces $f_1, f_2, f_3, ..., f_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$), let these k faces partition into m colour classes, then these m colour classes is called minimum colour classes, if whatever we try to partition these k faces into $m-1$ colour classes, there exist at least two adjacent faces belong to the same colour class.
- Definition 3.11. Let G be a graph of k common and non-common faces $f_1, f_2, f_3, ..., f_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$), let these k faces partition into m colour classes, then these m colour classes are called maximum colour classes, if whatever we try to partition these k faces into $m+1$ colour classes, there exist two colour classes, such that any two faces belong to these two colour classes, are not adjacent.
- Definition 3.12. Let G be a graph of common and non-common faces $f_1, f_2, f_3, ..., f_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$), these k faces partition into minimum m colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, if w^* be minimum number of proper

cliques $C_1^*, C_2^*, C_3^*, C_{w^*}^*$ of common and non-common faces $f_1^*, f_2^*, f_3^*, ..., f_{k^*}^*$ satisfies $w^* + 1 \le t_i + t_j$, for any two colour classes χ_i and χ_j, then the cliques $C_1^*, C_2^*, C_3^*, ..., C_{w^*}^*$ called standard of partition these k^* faces into minimum m colour classes, where t_i and t_j be number of times the colour classes χ_i and χ_j appears at these w^* cliques respectively, where $F^* \subseteq F$, $\Psi^* \subseteq \Psi$, or $k^* \le k$, $w^* \le w$, $F^* = \{f_1^*, f_2^*, f_3^*, ..., f_{k^*}^*\}$, $F = \{f_1, f_2, f_3, ..., f_k\}$, $\Psi^* = \{C_1^*, C_2^*, C_3^*, ..., C_{w^*}^*\}$, $\Psi = \{C_1, C_2, C_3, ..., C_w\}$, and $1 \le i < j \le m.$

- Proposition 3.13. Let G be a graph of k common and non-common faces $f_1, f_2, f_3, ..., f_k$, (common between proper cliques $C_1, C_2, C_3, ..., C_w$), with maximum clique number equal n, if these k faces partition into minimum m colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, and $f_1, f_2, f_3, ..., f_p$ are faces (common and non-common) belong to one colour class χ_i, if a_l be number of times the face f_l appear at these w cliques, where $1 \le l \le p$, $1 \le i \le m$, and t_i be number of times the colour class χ_i appear at these w cliques, then we have $t_i = a_1 + a_2 + a_3 + ... + a_p \le w.$

  Proof: Suppose $f_1, f_2, f_3, ..., f_p$ belong to one colour class χ_i and χ_i labelled by i, then each one of $f_1, f_2, f_3, ..., f_p$ labelled by i and the label i appear $a_1 + a_2 + a_3 + ... + a_p$ times at $C_1, C_2, C_3, ..., C_w$ then whatever we exchange partition of all faces (common and non-common) into m colour classes, and for the faces $f_1, f_2, f_3, ..., f_p$ belong to one colour class, any two faces of $f_1, f_2, f_3, ..., f_p$ are nonadjacent, then the label i cannot appear twice at any clique, if i appear once at any one of $C_1, C_2, C_3, ..., C_w$ then $t_i = a_1 + a_2 + a_3 + ... + a_p = w$, if i appear once at some of $C_1, C_2, C_3, ..., C_w$ then $t_i = a_1 + a_2 + a_3 + ... + a_p < w$, therefore we have $t_i = a_1 + a_2 + a_3 + ... + a_p \le w.$

- Proposition 3.14. Let G be a graph of k common faces $f_1, f_2, f_3, ..., f_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$), if these k faces partition into m colour

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classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, for all $1 \le j \le w$ then we have $\sum_{l=1}^{k} a_l = \sum_{j=1}^{w} d(C_j)$, where $1 \le l \le k$ and a_l is appearance of face f_l at w cliques.

Proof: Suppose these of k faces partition into m colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, labelled by $1,2,3,..., m,$ and let the colour class χ_i labelled by i. Since each clique C_j labelled by some number of $1,2,3,..., m,$ and this number equal $d(C_j)$, the number $d(C_j)$ equal number of faces of clique C_j, and each face f_l labelled by one label of $1,2,3,..., m,$ if $f_l \in \chi_i$, and the colour class χ_i contain only of f_l, the label i appears a_l times at w cliques, a_l means the face f_l common between a_l cliques. If the colour class χ_i consists only of faces f_l and f_r, the label i appears $a_l + a_r$ times at these w cliques and we can write $a_l + a_r = t_i$. If the colour class χ_i consists only of the faces f_l, f_r, f_s, the label i appears $a_l + a_r + a_s$ times at these w cliques, and we can write $a_l + a_r + a_s = t_i$, so from definition of clique number and definition of face common between some cliques, then we have $\sum_{l=1}^{k} a_l = \sum_{j=1}^{w} d(C_j)$.

- Proposition 3.15. Let G be a graph of k common and non-common faces $f_1, f_2, f_3, ..., f_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$), if these k faces partition into m colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, labelled by $1,2,3,..., m,$ and for all $1 \le j \le w$ then we have $\sum_{l=1}^{k_1} a_l + \sum_{l=1}^{k_2} b_l = \sum_{j=1}^{w} d(C_j)$, and $k_2 + \sum_{l=1}^{k_1} a_l = \sum_{j=1}^{w} d(C_j)$ where k_1 is number of common faces, and k_2 is number of non-common faces, a_l, b_l are appearance of common face and non-common face respectively at w cliques.

Proof: Using Proposition 3.14.if $k = k_1 + k_2$ where k_1 is number of common face, and k_2 is number of non-common faces, a_l, b_l are appearance of common face and non-common face at w cliques, then we have $\sum_{l=1}^{k_1} a_l + \sum_{l=1}^{k_2} b_l = \sum_{j=1}^{w} d(C_j)$. From definition of

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common face and non-common face at w cliques, we have $\sum_{l=1}^{k_2} b_l = k_2$, therefore $k_2 + \sum_{l=1}^{k_1} a_l = \sum_{j=1}^{w} d(C_j)$.

- Proposition 3.16. Let G be a graph of k common and non-common faces $f_1, f_2, f_3, \ldots, f_k$ (common between proper cliques $C_1, C_2, C_3, \ldots, C_w$), with maximum clique number equal n, and if these k faces partition into m colour classes $\chi_1, \chi_2, \chi_3, \ldots, \chi_m$, labelled by $1,2,3,\ldots,m$, and if t_i be number of times the colour class χ_i appear at these w cliques, where $1 \le i \le m$, and for all $1 \le j \le w$ then we have $\sum_{i=1}^{m} t_i = \sum_{l=1}^{k_1} a_l + \sum_{l=1}^{k_2} b_l$, and $\sum_{i=1}^{m} t_i = \sum_{l=1}^{k_1} a_l + \sum_{l=1}^{k_2} b_l = \sum_{j=1}^{w} d(C_j) \le wn.$

Proof: If for all $1 \le j \le w$, we have $d(C_j) = n$, then $\sum_{j=1}^{w} d(C_j) = wn.$ if for some $1 \le j \le w$, we have $d(C_j) < n$, then $\sum_{j=1}^{w} d(C_j) < wn.$ therefore we have $\sum_{j=1}^{w} d(C_j) \le wn.$ Using Proposition 3.15. We have $\sum_{l=1}^{k_1} a_l + \sum_{l=1}^{k_2} b_l = \sum_{j=1}^{w} d(C_j)$, if t_i be number of times the colour class χ_i appear at these w cliques, where $1 \le i \le m$, and since these k faces partition into m colour classes $\chi_1, \chi_2, \chi_3, \ldots, \chi_m$, labelled by $1,2,3,\ldots,m$, where $k_1 + k_2 = k$, then $\sum_{i=1}^{m} t_i = \sum_{l=1}^{k_1} a_l + \sum_{l=1}^{k_2} b_l$, and then we have $\sum_{i=1}^{m} t_i = \sum_{l=1}^{k_1} a_l + \sum_{l=1}^{k_2} b_l = \sum_{j=1}^{w} d(C_j) \le wn.$

- Proposition 3.17. Let G be a graph of k common and non-common faces $f_1, f_2, f_3, \ldots, f_k$ (common between proper cliques $C_1, C_2, C_3, \ldots, C_w$), with maximum clique number equal n, if these k faces partition into m minimum colour classes $\chi_1, \chi_2, \chi_3, \ldots, \chi_m$, labelled by $1,2,3,\ldots,m$, if we add only one non common face f_{k+1} without changing the maximum clique number n, then these $k+1$ faces partitioned into minimum colour classes also equal m.

Proof: Let these m minimum colour classes labeled by $1,2,3,\ldots,m,$ since after addition of a non-common face f_{k+1} the proper cliques $C_1, C_2, C_3, \ldots, C_k$ remain unchanged, except one clique, let that clique be C_j, and $d(C_j) = p$ before addition, after addition $d(C_j) = p+1,$ where $1 \le j \le w,\ 2 \le p \le n-1,$ let $C_j \equiv (1,2,3,\ldots,p)$ then after addition of the non-common face f_{k+1} we labelled C_j by $C_j \equiv (1,2,3,\ldots,p,q)$ where $p+1 \le q \le m,$ and the face f_{k+1} belongs to colour class χ_q where $p+1 \le q \le m,$ and since before we add the face f_{k+1}, $d(C_j) \le n-1$ then the face f_{k+1} can belongs to one of the following colour classes $\chi_{p+1}, \chi_{p+2}, \chi_{p+3}, \ldots, \chi_m$. Then these $k+1$ common and non-common faces partitioned into m colour classes.

- Corollary 3.18. Let G be a graph of k common and non-common faces $f_1, f_2, f_3, \ldots, f_k$ (common between proper cliques $C_1, C_2, C_3, \ldots, C_w$), with maximum clique number equal n, if these k faces partition into m minimum colour classes $\chi_1, \chi_2, \chi_3, \ldots, \chi_m$, labelled by $1,2,3,\ldots,m,$ if the face f_{k+1} is non common face, and we remove only the face f_{k+1} without changing the maximum clique number n, then these $k-1$ faces partitioned into minimum colour classes also equal m.

Proof: Using Proposition 3.17. if we add a non-common face f_{k+1} without changingthe maximum clique number n, then these $k+1$ faces partitioned into minimum colour classes also equal m. If we remove the non-common face f_{k+1}, then the maximum clique number remains n, then these k faces was partitioned into minimum colour classes also equal m. Hence removal of a non-common face without changing the maximum clique number n, doesn't change number of minimum colour classes.

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4. Method of Finding the Minimum Number of Colour Classes:

This technique is same as method of finding minimum number of colour classes for edge colouring [13], and vertex colouring [14]. In this section we give some examples to explain this method, and introduce some results related to this method.

- Proposition 4.1. Let G be a graph of k faces $f_1, f_2, f_3,..., f_k$ and $C_1, C_2, C_3,..., C_w$ are proper cliques of these k faces, with maximum cliques number equal n, suppose all faces partition into m colour classes $\chi_1, \chi_2, \chi_3,..., \chi_m$, using the following method: In first step we try to find q_0 colour classes, which is maximum number of colour classes each has appearance equal w at $C_1, C_2, C_3,..., C_k$ cliques, such that each colour class consists only of common faces. In second step we try to find q_1 colour classes, which is maximum number of colour classes each has appearance equal $w-1$ at $C_1, C_2, C_3,..., C_k$ cliques, such that each colour class consists only of common faces. In third step we try to find q_2 colour classes, which is maximum number of colour classes each has appearance equal $w-2$ at $C_1, C_2, C_3,..., C_k$ cliques, such that each colour class consists only of common faces. We continue in same process, at last step we try to find q_s colour classes, which is maximum number of colour classes each has appearance equal $w-s$ at $C_1, C_2, C_3,..., C_k$ cliques, such that it is impossible to be q_s+1 colour classes, and such that each colour class consists only of common faces. After last step if all common faces partition into colour classes m, where $m = q_0 + q_1 + q_2 + ... + q_s$, then m is minimum number of colour classes.

  Proof: Suppose using only step I we partition all common faces into m colour classes and neglect non common faces, due Proposition 3.17. , this means $m = q_0$.

  Using Proposition 3.16. We have $\sum_{i=1}^{m} t_i = \sum_{l=1}^{k_1} a_l + \sum_{l=1}^{k_2} b_l = \sum_{j=1}^{w} d(C_j) \le wn,$ and since

$n \le m,$ we can write $\sum_{i=1}^{m} t_i = wq_0 = wm = \sum_{j=1}^{w} d(C_j) \le wn,$ it a contradiction if $m < n,$ then we have $n = m,$ therefore m is minimum number of colour classes. This means the result holds using only step I.

Suppose using only step I and step II we partition all common faces into m colour classes and neglect non common faces, this means $m = q_0 + q_1.$ Using Proposition 3.16. We can write $\sum_{i=1}^{m} t_i = wq_0 + (w-1)q_1 + r = \sum_{j=1}^{w} d(C_j) \le wn,$ where $0 \le r$, and r is number of non-common faces, from Proposition 3.17. after addition of these r non-common faces, number of colour classes remain $m = q_0 + q_1.$ Suppose all common faces partition into $m-1$ colour classes instate of m colour classes, where $m-1 = q_0 + q_1 - 1,$ then either there is $q_0 + x_0$ colour classes has appearance equal w or there is there is $q_1 + x_1$ colour classes has appearance equal $w-1,$ both x_0, x_1 are integers $0 \le x_0,$ $0 \le x_1$ not both of them equal zero. Then there at least $q_0 + 1$ colour classes each has appearance equal w at $C_1, C_2, C_3, ..., C_k$ cliques, or there at least $q_1 + 1$ colour classes each has appearance equal $w-1$ at $C_1, C_2, C_3, ..., C_k$ cliques, it is a contradiction if all common faces partition into $m-1$ colour classes. Then m is minimum number of colour classes. This means the result holds using only step I and step II.

Suppose using only step I, stepII and step III we partition all common faces into m colour classes and neglect non common faces, this means $m = q_0 + q_1 + q_2.$ Using Proposition 3.16. We can write $\sum_{i=1}^{m} t_i = wq_0 + (w-1)q_1 + (w-2)q_2 + r = \sum_{j=1}^{w} d(C_j) \le wn,$ where $0 \le r$, and r is number of non-common faces. From Proposition 3.17. after addition of these r non-common faces, number of colour classes remain $m = q_0 + q_1 + q_2.$ Suppose all common faces partition into $m-1$ colour classes instate of m colour classes, where $m-1=q_0+q_1+q_2-1,$ then either there is q_0+x_0 colour classes has appearance equal w or there is there is q_1+x_1 colour classes has appearance equal $w-1,$ or there is q_2+x_2 colour classes has appearance equal $w-2,$ all of x_0,x_1,x_2 are integers $0\le x_0,\ 0\le x_1,\ 0\le x_2,$ not all of them equal zero. Then there at least q_0+1 colour classes each has appearance equal w at $C_1,C_2,C_3,...,C_k$ cliques, or there at least q_1+1 colour classes each has appearance equal $w-1$ at $C_1,C_2,C_3,...,C_k$ cliques, or there at least

q_2+1 colour classes each has appearance equal $w-2$ at $C_1,C_2,C_3,...,C_k$ cliques, it is a contradiction if all common faces partition into $m-1$ colour classes. Then m is minimum number of colour classes. This means the result holds using only step I, step II and step III. Continue in this process if $m=q_0+q_1+q_2+...+q_s,$ then m is minimum number of colour classes. This means the result holds using all steps.

- Remark 4.2. In this paper and coming papers we called method in Proposition 4.1. Method of finding minimum number of colour classes.
- Proposition 4.3. Let G be a graph of k faces $f_1, f_2, f_3,..., f_k$ and $C_1,C_2,C_3,...,C_w$ are proper cliques of these k faces, with maximum clique number equal $n,$ where $2<n,$ and there is a proper clique with clique number equal two. If all faces (common faces and non-common faces) partitioned into minimum m colour classes, then the number of colour classes appear w times at $C_1,C_2,C_3,...,C_w$ not more than two.

Proof: Suppose the number of colour classes appear times w at $C_1,C_2,C_3,...,C_w$ equal three or more than three, let be three colour classes each appear times w at $C_1,C_2,C_3,...,C_w,$ from definition of colour class, each one of χ_1,χ_2,χ_3 appear at all of cliques then there is no clique with clique number less than three, a contradiction since there is one clique with clique number equal two, it is also a contradiction for more than three colour classes each appears w times at

$C_1, C_2, C_3, ..., C_w,$ therefore the number of colour classes appear w times at $C_1, C_2, C_3, ..., C_w$ not more than two.

Next proposition is generalization of Proposition 4.3., the proof is same as Proposition 4.3.

- Proposition 4.4. Let G be a graph of k faces $f_1, f_2, f_3, ..., f_k$ and $C_1, C_2, C_3, ..., C_w$ are proper cliques of these k faces, with maximum clique number equal $n,$ where $q < n,$ and there is a proper clique with clique number equal $q.$ If all faces (common faces and non-common faces) partitioned into minimum m colour classes, then the number of colour classes appear w times at $C_1, C_2, C_3, ..., C_w$ not more than $q.$
- Proposition 4.5. Let G be a graph of k faces $f_1, f_2, f_3, ..., f_k$ and $C_1, C_2, C_3, ..., C_w$ are proper cliques of these k faces, with maximum clique number equal $n,$ let these faces be partitioned into m colour classes, where $m = n + 1,$ and there is no colour class appears w times at $C_1, C_2, C_3, ..., C_w$, and there is n colour classes each appears $w - 1$ times, and there is a colour class appears r times, where $2 \le r \le w - 2,$ if $n(w-1) + r = \sum_{j=1}^{w} d(C_j),$ then m is minimum number of colour classes.

Proof: To show m is minimum number of colour classes, suppose these k common faces partition into $m - 1$ colour classes, then there exists at least one colour class χ_i such that $t_i = w + 1,$ where $1 \le i \le m - 1,$ and then the colour class have two adjacent faces belong to the colour class $\chi_i.$ Therefore m is minimum number of colour classes.

- Corollary 4.6. Let G be a graph of k faces $f_1, f_2, f_3, ..., f_k$ and $C_1, C_2, C_3, ..., C_w$ are proper cliques of these k faces, with maximum clique number equal $k = k_1 + k_2,$ where k_1 number of common faces, and k_2 number of non-common faces, if these k_1 common faces partition into m colour classes, where $m = n + 1,$ such there is no colour class appears w times at $C_1, C_2, C_3, ..., C_w$, and there is n colour classes each appears $w-1$ times, and there is a colour class appears r times, where $2 \le r \le w-2$, and $n(w-1)+r=\sum_{l=1}^{k_1} a_l$, where a_l is appearance of face f_l and $1 \le l \le k_1$. Then these k faces partition into m is minimum number of colour classes.

Proof: Using Proposition 4.5 these k_1 common faces partition into m minimum colour classes, since $k=k_1+k_2$, and k_2 are non-common faces, Using Proposition3.17. After addition of k_2 non-common faces, then these k faces partition into m is minimum number of colour classes.

5. Common Face and Non Common Face Related to Proper Vertex:

In this section we introduce the concept of common face and concept of non-common face related to proper vertex separates faces. In this section two adjacent faces means they have common vertex. In this section we introduce some results without proofs, for any proof you can see an analog result in section three.

- Definition 5.1. degree of a vertex v w.r.t. faces $f_1, f_2, f_3, ..., f_n$, is the number of faces separated by v, or we can say number of faces incident by v. We use the notation $d^*(v)=n$ w.r.t. to faces
- Remark 5.2. Let v be a vertex w.r.t. to faces and with degree equal n, such that v separate by faces $f_1, f_2, f_3, ..., f_n$, then we use the notation $v \equiv (f_1, f_2, f_3, ..., f_n)$, or $v=\{f_1, f_2, f_3, ..., f_n\}$.
- Definition 5.3. Let v_1 be vertex separates faces $\{f_1, f_2, f_3, ..., f_t\}$, then v_1 is called proper vertex separated faces $\{f_1, f_2, f_3, ..., f_t\}$, or v_1 is called proper vertex of faces $\{f_1, f_2, f_3, ..., f_t\}$, if there is no a vertex v_2 separates faces $\{f_1^*, f_2^*, f_3^*, ..., f_s^*\}$ such that

$\{f_1, f_2, f_3,..., f_t\} \subset \{f_1^*, f_2^*, f_3^*,..., f_s^*\}$, $t < s$, and $v_1 \subset v_2$, and for all i and some j we have $f_i = f_j^*$, where $v_1 = \{f_1, f_2, f_3,..., f_t\}$, $v_2 = \{f_1^*, f_2^*, f_3^*,..., f_s^*\}$, $1 \le i \le t$ and $1 \le j \le s$.

- Example 5.4. Let v_1 be vertex separates faces $\{f_1, f_2, f_3, f_4\}$, i.e $v_1 \equiv (f_1, f_2, f_3, f_4)$, and v_2 be vertex separates faces $\{f_1, f_2, f_3\}$, i.e $v_2 \equiv (f_1, f_2, f_3)$, then we can say v_1 is proper vertex of faces $\{f_1, f_2, f_3, f_4\}$, and v_2 is not proper vertex of faces $\{f_1, f_2, f_3\}$.
- Definition 5.5. In a graph G if each of v_1 and v_2 be a proper vertex of the face f, then f is called common between two vertices v_1 and v_2, where $v_1 \ne v_2$. Let each of $v_1, v_2, v_3,..., v_t$ is a proper vertex of the face f, then the face f is called common face between t proper vertices $v_1, v_2, v_3,..., v_t$ if each of $v_i \ne v_j$, for all i, j where $1 \le i < j \le t$.
- Definition 5.6. In a graph G the face f is called non common between any number of vertices, if there is only one proper vertex separates all faces adjacent to f or f is isolated face. This means a non-common face belongs to only one vertex.
- Definition 5.7. Let C_1 be clique of the faces $f_1, f_2, f_3,..., f_t$, then C_1 is called isolate vertex of faces $f_1, f_2, f_3,..., f_t$ if each one of $f_1, f_2, f_3,..., f_t$ is a non-common face.
- Definition 5.8. Let G be a graph of k common and non-common faces $f_1, f_2, f_3,..., f_k$ (common between proper vertices $v_1, v_2, v_3,..., v_t$), and these k faces partition into m colour classes, then these m colour classes is called minimum colour classes, if whatever we try to partition these k faces into $m-1$ colour classes, there exist at least two adjacent faces belong to the same colour class.
- Definition 5.9. Let G be a graph of k common and non-common faces $f_1, f_2, f_3,..., f_k$ (common between proper vertices $v_1, v_2, v_3,..., v_t$), and these k faces partition into m colour classes, then these m colour classes are called maximum colour classes, if whatever we try to partition these k faces into $m+1$ colour classes, there exist two

colour classes, such that any two faces belong to these two colour classes, are not adjacent.

- Definition 5.10. Let G be a graph of common and non-common faces $f_1, f_2, f_3,..., f_k$ (common between proper vertices $v_1, v_2, v_3,..., v_w$), and these k faces partition into minimum m colour classes $\chi_1, \chi_2, \chi_3,..., \chi_m$, if w^* be minimum number of proper vertices $v_1^*, v_2^*, v_3^*,..., v_{w^*}^*$ of common and non-common faces $f_1^*, f_2^*, f_3^*,..., f_{k^*}^*$ satisfies $w^* + 1 \le t_i + t_j$, for any two colour classes χ_i and χ_j, then the vertices $v_1^*, v_2^*, v_3^*,..., v_{w^*}^*$ called standard of partition these k^* faces into minimum m colour classes, where t_i and t_j be number of times the colour classes χ_i and χ_j appears at these w^* vertices respectively, where $F^* \subseteq F$, $\Psi^* \subseteq \Psi$, or $k^* \le k$, $w^* \le w$, $F^* = \{f_1^*, f_2^*, f_3^*,..., f_{k^*}^*\}$, $F = \{f_1, f_2, f_3,..., f_k\}$, $\Psi^* = \{v_1^*, v_2^*, v_3^*,..., v_{w^*}^*\}$, $\Psi = \{v_1, v_2, v_3,..., v_w\}$, and $1 \le i < j \le m$.
- Proposition 5.11. Let G be a graph of k common and non-common faces $f_1, f_2, f_3,..., f_k$, (common between proper vertices $v_1, v_2, v_3,..., v_w$), with maximum degree of a vertex w.r.t. faces equal n, if these k faces partition into minimum m colour classes $\chi_1, \chi_2, \chi_3,..., \chi_m$, and $f_1, f_2, f_3,..., f_p$ are faces (common and non-common) belong to one colour class χ_i, if a_l be number of times the face f_l appear at these w vertices, where $1 \le l \le p, 1 \le i \le m$, and t_i be number of times the colour class χ_i appear at these w vertices, then we have $t_i = a_1 + a_2 + a_3 + ... + a_p \le w.$
- Proposition 5.12. Let G be a graph of k common faces $f_1, f_2, f_3,..., f_k$ (common between proper vertices $v_1, v_2, v_3,..., v_w$), if these k faces partition into m colour classes $\chi_1, \chi_2, \chi_3,..., \chi_m$, for all $1 \le j \le w$ then we have $\sum_{l=1}^{k} a_l = \sum_{j=1}^{w} d(C_j)$, where $1 \le l \le k$ and a_l is appearance of face f_l at w vertices.

- Proposition 5.13. Let G be a graph of k common and non-common faces $f_1, f_2, f_3,..., f_k$ (common between proper vertices $v_1, v_2, v_3,..., v_w$), if these k faces partition into m colour classes $\chi_1, \chi_2, \chi_3,..., \chi_m$, labelled by $1,2,3,..., m,$ and for all $1 \le j \le w$ then we have $\sum_{l=1}^{k_1} a_l + \sum_{l=1}^{k_2} b_l = \sum_{j=1}^{w} d(v_j),$ and $k_2 + \sum_{l=1}^{k_1} a_l = \sum_{j=1}^{w} d(v_j),$ where k_1 is number of common faces, and k_2 is number of non-common faces, a_l, b_l are appearance of common face and non-common face respectively at w vertices.
- Proposition 5.14. Let G be a graph of k common and non-common faces $f_1, f_2, f_3,..., f_k$ (common between proper vertices $v_1, v_2, v_3,..., v_w$), with maximum degree of a vertex w.r.t. faces equal $n,$ and if these k faces partition into m colour classes $\chi_1, \chi_2, \chi_3,..., \chi_m$, labelled by $1,2,3,..., m,$ and if t_i be number of times the colour class χ_i appear at these w vertices, where $1 \le i \le m,$ and for all $1 \le j \le w$ then we have $\sum_{i=1}^{m} t_i = \sum_{l=1}^{k_1} a_l + \sum_{l=1}^{k_2} b_l,$ and $\sum_{i=1}^{m} t_i = \sum_{l=1}^{k_1} a_l + \sum_{l=1}^{k_2} b_l = \sum_{j=1}^{w} d(v_j) \le wn.$
- Proposition 5.15. Let G be a graph of k common and non-common faces $f_1, f_2, f_3,..., f_k$ (common between proper vertices $v_1, v_2, v_3,..., v_w$), with maximum degree of a vertex w.r.t. faces equal $n,$ if these k faces partition into m minimum colour classes $\chi_1, \chi_2, \chi_3,..., \chi_m$, labelled by $1,2,3,..., m,$ if we add only one non common face f_{k+1} without changing the maximum degree of a vertex w.r.t. faces $n,$ then these $k+1$ faces partitioned into minimum colour classes also equal m.
- Corollary 5.16. Let G be a graph of k common and non-common faces $f_1, f_2, f_3,..., f_k$ (common between proper vertices $v_1, v_2, v_3,..., v_w$), with maximum degree of a vertex w.r.t. faces equal $n,$ if these k faces partition into m minimum colour classes $\chi_1, \chi_2, \chi_3,..., \chi_m$, labelled by $1,2,3,..., m,$ if the face f_{k+1} is non common

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face, and we remove only the face f_{k+1} without changing the maximum clique number n, then these k faces partitioned into minimum colour classes also equal m.

- Proposition 5.17. Let G be a graph of k faces $f_1, f_2, f_3,..., f_k$ and $v_1, v_2, v_3,..., v_w$ are proper vertices of these k faces, with maximum degree of a vertex w.r.t. faces equal n, suppose all faces partition into m colour classes $\chi_1, \chi_2, \chi_3,..., \chi_m$, using the following method: In first step we try to find q_0 colour classes, which is maximum number of colour classes each has appearance equal w at $v_1, v_2, v_3,..., v_w$ vertices, such that each colour class consists only of common faces. In second step we try to find q_1 colour classes, which is maximum number of colour classes each has appearance equal $w-1$ at $v_1, v_2, v_3,..., v_w$ vertices, such that each colour class consists only of common faces. In third step we try to find q_2 colour classes, which is maximum number of colour classes each has appearance equal $w-2$ at $v_1, v_2, v_3,..., v_w$ vertices, such that each colour class consists only of common faces. We continue in same process, at last step we try to find q_s colour classes, which is maximum number of colour classes each has appearance equal $w-s$ at $v_1, v_2, v_3,..., v_w$ vertices, such that it is impossible to be q_s+1 colour classes, and such that each colour class consists only of common faces. After last step if all common faces partition into colour classes m, where $m = q_0 + q_1 + q_2 + ... + q_s$, then m is minimum number of colour classes.

Multiple Vertices and Multiple Faces:

In this section we introduce some definitions and results about multiple vertices and multiple faces.

- Definition 6.1. Let G be a graph of k common and non-common vertices $v_1, v_2, v_3,..., v_k$ (common between proper cliques $C_1, C_2, C_3,..., C_w$), if v_1 is common between cliques $C_1,C_2,C_3,...,C_r,$ and v_2 is common between cliques $C_1,C_2,C_3,...,C_r,$ then vertices v_1,v_2 are called multiple (congruent) at r cliques.

- Definition6. 2. Let G be a graph of k vertices $v_1,v_2,v_3,...,v_k$ (common between proper cliques $C_1,C_2,C_3,...,C_w$), if v_1 is common between cliques $C_1,C_2,C_3,...,C_r,C_{r+1},C_{r+2},...,C_s$ and v_2 is common between cliques $C_1,C_2,C_3,...,C_r,C_{s+1},C_{s+2},...,C_t,$ such that $\{C_{r+1},C_{r+2},C_{r+3},...,C_s\}\bigcap\{C_{s+1},C_{s+2},C_{s+3},...,C_t\}=\phi,$ then the vertices v_1,v_2 are called semi multiple at r cliques.
- Definition 6.3. Let G be a graph of k common and non-common faces $f_1, f_2, f_3,..., f_k$ (common between proper cliques $C_1,C_2,C_3,...,C_w$), if f_1 is common face between cliques $C_1,C_2,C_3,...,C_r,$ and f_2 is common face between cliques $C_1,C_2,C_3,...,C_r,$ then the faces f_1, f_2 are called multiple(congruent) faces at r cliques.
- Definition 6.4. Let G be a graph of k faces $f_1, f_2, f_3,..., f_k$ (common between proper cliques $C_1,C_2,C_3,...,C_w$), if f_1 is common face between cliques $C_1,C_2,C_3,...,C_r,C_{r+1},C_{r+2},...,C_s$ and f_2 is common face between cliques $C_1,C_2,C_3,...,C_r,C_{s+1},C_{s+2},...,C_t,$ such that $\{C_{r+1},C_{r+2},C_{r+3},...,C_s\}\bigcap\{C_{s+1},C_{s+2},C_{s+3},...,C_t\}=\phi,$ then faces f_1, f_2 are called semi multiple at r cliques.
- Proposition 6.5. and Proposition 6. 6. Are analog, so no need to prove Proposition 6.6.

Proposition 6.5. Let G be a graph of k vertices $v_1,v_2,v_3,...,v_k$ and $C_1,C_2,C_3,...,C_w$ are proper cliques of these k vertices, if w is odd and $a_1=a_2=\frac{w+3}{2},$ then vertices v_1,v_2 are semi multiple at least at 3 cliques, where a_1,a_2 is appearance of v_1,v_2 at $C_1,C_2,C_3,...,C_w$ respectively.

Proof: To arrange vertices v_1, v_2 to be semi multiple at minimum number of cliques, so v_1 appear at $C_1, C_2, C_3, ..., C_{\frac{w+3}{2}}$ and v_1 appear at $C_w, C_{w-1}, C_{w-2}, ..., C_{\frac{w-1}{2}}$. the vertices v_1, v_2 are have repeated appearance at cliques $C_{\frac{w-1}{2}}, C_{\frac{w+1}{2}}, C_{\frac{w+3}{2}}$. Therefore vertices v_1, v_2 are semi multiple least at 3 cliques.

- Proposition 6.6. Let G be a graph of k faces $f_1, f_2, f_3, ..., f_k$ and $C_1, C_2, C_3, ..., C_w$ are proper cliques of these k faces, if w is odd and $a_1 = a_2 = \frac{w+3}{2}$, then faces f_1, f_2 are semi multiple at least at 3 cliques, where a_1, a_2 is appearance of f_1, f_2 at $C_1, C_2, C_3, ..., C_w$ respectively.

For proofs of Proposition 6.7. and Proposition 6.8. See Proposition 6.5 paper [14] as an analog proposition.

- Proposition 6.7. Let G be a graph of k vertices $v_1, v_2, v_3, ..., v_k$ and $C_1, C_2, C_3, ..., C_w$ are proper cliques of these k vertices, with maximum clique number equal n, suppose these k vertices partitioned into minimum colour classes equal $n+1$, suppose each colour class appears $w-1$ times or all colour classes (except one colour class) appears $w-1$ times at $C_1, C_2, C_3, ..., C_w$. Then $xw \le n+x$ is condition if these k vertices partitioned into minimum number colour classes equal $n+x$. where $x \le n-1$.
- Proposition 6.8. Let G be a graph of k faces $f_1, f_2, f_3, ..., f_k$ and $C_1, C_2, C_3, ..., C_w$ are proper cliques of these k faces, with maximum clique number equal n, suppose these k faces partitioned into minimum colour classes equal $n+1$, suppose each colour class appears $w-1$ times or all colour classes (except one colour class) appears $w-1$ times at $C_1, C_2, C_3, ..., C_w$. Then $xw \le n+x$ is condition if these k edges partitioned into minimum number colour classes equal $n+x$. where $x \le n-1$.

Graph of k common and non-common faces $f_1, f_2, f_3, ..., f_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$), with maximum clique number equal $n,$

- Proposition 6.9. If v_1 is a vertex common between cliques $C_1, C_2, C_3, ..., C_r,$ each clique of degree $n,$ and v_1 is adjacent to l vertices. Then $l = (n-1)r$ if and only if v_1 and any one of these l vertices is not semi multiple at r cliques.

  Proof: First suppose v_1 and any one of these l vertices is not semi multiple at r cliques. We label v_1 by 1 and these l vertices $v_2, v_3, v_4, ..., v_{l+1}$ by $2,3,4,...,l+1$ respectively. Let us label the cliques $C_1, C_2, C_3, ..., C_r$ as follows: $C_1 \equiv (1,2,3,..,n),$ $C_2 \equiv (1, n+1, n+2, n+3,..,2n-1),$ $C_3 \equiv (1,2n,2n+1,2n+2,..,3n-2),$ $C_{r-1} \equiv (1,(r-2)n-r+4,(r-2)n-r+5,(r-2)n-r+6,..,(r-1)n-(r-2)),$ $C_r \equiv (1,(r-1)n-(r-3),(r-1)n-(r-4),..,rn-(r-1)),$ then $l+1 = rn-(r-1),$ and then $l = (n-1)r.$ If $l < (n-1)r$ then v_1 adjacent to number of vertices less than $l,$ a contradiction. If $(n-1)r < l$ then there is a clique with clique number greater than $n,$ a contradiction.

  Conversely suppose $l = (n-1)r,$ since v_1 is common between cliques $C_1, C_2, C_3, ..., C_r,$ each clique of degree $n,$ and v_1 is adjacent to l vertices, then $C_i \cap C_j = v_1,$ where $1 \le i < j \le r,$ then v_1 and any one of these l vertices is not semi multiple at r cliques.

- Proposition 6.10. If e_1 is an edge common between vertices v_1, v_2 each vertex of degree $n,$ and e_1 is adjacent to l edges. Then $l = 2(n-1)$ if and only if e_1 and any one of these l edges is not multiple at these two vertices.

- Proposition. 6.11. If f_1 is common face between cliques $C_1, C_2, C_3, ..., C_r,$ each clique of degree $n,$ and f_1 is adjacent to l faces. Then $l = (n-1)r$ if and only if f_1 and any one of these l faces is not semi multiple at r cliques.

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Each of Proposition 6.12., Proposition 6.13. and Proposition 6.14. is the generalization of Proposition 6.12., Proposition 6.13. and Proposition 6.14respectively.

- Proposition 6.12. If v_1 is a vertex common between cliques $C_1, C_2, C_3, ..., C_r$, each clique of degree n, and v_1 is adjacent to l vertices. Then $l = \sum_{j=1}^{r} d(C_j) - r$ if and only if v_1 and any one of these l vertices is not semi multiple at r cliques. If $l < \sum_{j=1}^{r} d(C_j) - r$ then v_1 and at least one vertex of these l vertices, is semi multiple at r cliques.
- Proposition 6.13. If e_1 is an edge common between vertices v_1, v_2 each vertex of degree n, and e_1 is adjacent to l edges. Then $l = d(v_1) + d(v_2) - 2$ if and only if e_1 and any one of these l edges are not multiple between these two vertices. If $l < d(v_1) + d(v_2) - 2$ then e_1 is multiple with at least one of these l edges
- Proposition. 614. If f_1 is common face between cliques $C_1, C_2, C_3, ..., C_r$, each clique of degree n, and f_1 is adjacent to l faces. Then $l = \sum_{j=1}^{r} d(C_j) - r$ if and only if f_1 and any one of these l faces is not semi multiple at r cliques. If $l < \sum_{j=1}^{r} d(C_j) - r$ then f_1 and at least one face of these l faces, is semi multiple at r cliques.

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