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A Variable-Order Fractional Financial Model with State-Dependent Memory

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Abstract

This study presents a novel state-dependent fractional-order financial model that advances the modeling of memory effects in dynamic economic systems. By allowing the memory order $q(t)$ to adapt continuously as a function of the internal state variable—specifically, the investment demand $y(t)$—the model captures the essential nonlinear feedback between market sentiment, memory depth, and macroeconomic behavior. The fractional framework is based on the Caputo-Fabrizio operator, chosen for its smooth, non-singular kernel, which ensures physical consistency and superior numerical tractability.

A hyperbolic tangent structure is employed to define the adaptive memory function, enabling bounded, smooth, and symmetric evolution of $q(t)$ within a realistic economic range. This design reflects empirically observed financial phenomena: heightened responsiveness during periods of uncertainty (short memory) and persistent inertia during stable or optimistic regimes (long memory). Numerical simulations demonstrate the model's ability to replicate key financial features—such as damped oscillations, delayed stabilization, and variable sensitivity—under changing market conditions.

Furthermore, the model is validated against real data from the 2008 global financial crisis, illustrating its empirical relevance and practical forecasting potential. By integrating internal memory regulation into the core of financial system dynamics, this work contributes a flexible and realistic tool for modern economic analysis, with applications in policy modeling, risk evaluation, and adaptive control of financial systems.

Keywords: Adaptive Fractional-Order Modeling, Variable-Order Derivatives,

Caputo-Fabrizio Operator, Investment Driven Memory, Interest Rate Dynamics, State-Dependent Memory, Nonlinear Financial Systems, Economic Stability, Fractional Differential Equations.

1. Introduction

Fractional calculus has emerged as a robust mathematical framework for modeling complex systems with memory, hereditary properties, and non-local interactions. These features are particularly relevant in financial markets, where dynamics are influenced not only by present conditions but also by the cumulative effects of historical behaviors, policy inertia, and macroeconomic expectations. Unlike classical integer-order models, fractional-order systems incorporate memory kernels, enabling the representation of long-term dependencies often observed in investment cycles, interest rate dynamics, and economic shocks [19, 1, 20].

Over the last decade, the application of fractional differential equations (FDEs) in economic modeling has attracted considerable attention. For instance, Li et al. [7] explored a chaotic financial system with variable-order memory and demonstrated how fractional dynamics could alter system sensitivity and stability. Herrera et al. [8] developed numerical schemes for solving such systems using both Caputo and Caputo-Fabrizio derivatives, highlighting their effectiveness in capturing fading memory in real-world economic behaviors.

In particular, the Caputo-Fabrizio (CF) derivative has received growing interest due to its non-singular exponential kernel, which ensures smoother memory decay and improved numerical stability [2]. It has been applied to financial models dealing with inflation, credit dynamics, and portfolio synchronization, as demonstrated by Aboodh and Alsmadi [9] and Mahdy [3]. These studies confirmed the CF operator's potential for modeling memory-based transitions and damped oscillatory behavior in financial systems.

Several extensions to classical memory structures have also been proposed, including the use of Mittag Leffler kernels and generalized non-local operators to better reflect empirical patterns of memory decay [10]. These formulations are particularly useful in modeling phenomena such as persistent volatility, delayed investment cycles, and slow policy feedback. Reviews such as Baleanu et al. [4] provide further evidence for the relevance of fractional models in capturing the complexity of economic behavior.

However, a key limitation of many existing approaches is the assumption that the memory order $q(t)$ is either constant or a function of time alone. This assumption neglects the endogenous nature of memory in economic systems—where the persistence of historical influence often depends on internal state variables such as interest rates or investment demand. As shown in behavioral macroeconomic studies, memory can vary across market conditions: it becomes shorter during crises and longer during stable periods [17, 5].

In response to this gap, we propose a variable-order financial model where $q(t)$ is driven by internal economic activity. Specifically, we consider two adaptive formulations: one where memory evolves with investment demand $y(t)$, and another where it is linked to interest rate $x(t)$. These two state variables reflect distinct economic mechanisms—market-driven behavior and policy-driven influence—and are critical for understanding memory dynamics. The model is implemented using Caputo-Fabrizio derivatives and validated through simulations and real data from the 2008 global financial crisis, demonstrating its applicability to scenarios with shifting market memory and regulatory responses [18, 16, 13].

2. Preliminaries, Algorithms and Modeling

This section introduces the mathematical foundations, numerical strategy, and formulation of the adaptive financial model used in this study. We begin by defining the Caputo-Fabrizio fractional derivative and its variable-order extension, followed by a description of the numerical method adapted for simulation. Finally, we present the nonlinear financial system structure governed by state-dependent memory.

2.1 Caputo-Fabrizio Fractional Derivative:

The Caputo-Fabrizio (CF) derivative, first proposed in [2], uses an exponential kernel instead of the classical singular kernel. This property allows for smooth memory fading and improved numerical behavior in economic systems.

Definition 1 (Caputo-Fabrizio Derivative [2]). *Let* $f \in C^1[0, T]$. *The CF fractional derivative of order* $q \in (0, 1)$ *is defined as:*

$$ {}_{0}^{CF}D^{q} f(t) = \frac{1}{1-q} \int_{0}^{t} f'(\tau) \exp\left(-\frac{q}{1-q}(t-\tau)\right) d\tau. \qquad (1)$$

Unlike Caputo or Riemann–Liouville derivatives, the CF operator avoids singularities and is better suited to model physical systems with finite memory horizons [8, 4].

2.2 Variable-Order Caputo-Fabrizio Derivative:

In adaptive systems, memory depth should evolve with internal states. Thus, we extend the CF operator to variable-order form:

Definition 2 (Variable-Order CF Derivative [7]). *Let* $q(t) \in (0, 1)$. *The variable-order Caputo-Fabrizio derivative of a differentiable function* $f(t)$ *is:*

$$ {}_{0}^{CF}D^{q(t)} f(t) = \frac{1}{1-q(t)} \int_{0}^{t} f'(\tau) \exp\left(-\frac{q(t)}{1-q(t)}(t-\tau)\right) d\tau. \qquad (2)$$

This form maintains smoothness and supports state-based adaptation of memory, as used in economic and biological systems [8].

2.3 Numerical Approximation Scheme:

Due to the non-locality of the CF derivative, we employ a modified Euler-type method that accommodates the changing memory order at each time step. The domain is discretized as $t_n = nh$, with constant step size h. The approximation of the CF derivative at t_n is:

$$ {}_{0}^{CF}D^{q(t_n)} f(t_n) \approx \frac{1}{1-q(t_n)} \sum_{k=0}^{n-1} (f(t_{k+1}) - f(t_k)) \exp\left(-\frac{q(t_n)}{1-q(t_n)}(t_n - t_k)\right). \qquad (3)$$

This method ensures stability and accuracy when applied to systems with variable-order memory kernels [3].

Algorithm 1 Euler-Based Method for Variable-Order CF System
1: **Initialize all state variables:** $x(0), y(0), z(0), u(0)$
2: **for** $n = 0$ to $N - 1$ **do**
3: **Compute memory order:** $q(t_n) = q_0 + \mu \cdot \tanh(\lambda \cdot s(t_n))$
4: **Apply CF-based update rule to each variable**
5: **end for**

2.4 Financial System Formulation:

The model is based on a nonlinear system of four state variables:

- $x(t)$: interest rate,
- $y(t)$: investment demand,
- $z(t)$: price index,
- $u(t)$: average profit margin.

The governing equations are adapted from the fractional-order financial model presented by Malaikah and Al-Abdali [6] and extended here using a variable memory order:

$$\begin{cases} {}_0^{CF}D^{q(t)}x(t) = z(t) + (y(t) - a)x(t) + u(t), \\ {}_0^{CF}D^{q(t)}y(t) = 2 - by(t) - x(t)^2, \\ {}_0^{CF}D^{q(t)}z(t) = x(t)y(t) - x(t) - cz(t), \\ {}_0^{CF}D^{q(t)}u(t) = -dx(t)y(t) - gu(t), \end{cases} \tag{4}$$

Where:

- a: saving rate,
- b: cost per investment,
- c: market elasticity,
- $d, g > 0$: interaction and dissipation parameters.

The order function $q(t)$ is defined based on a chosen internal variable $s(t) \in \{x(t), y(t)\}$, allowing flexibility in modeling monetary or demand-driven memory effects.

3. State-Dependent Variable-Order

This section introduces a state-dependent formulation of the fractional order, where memory adapts according to internal economic variables such as investment demand or interest rates [8, 7]. To capture this adaptive behavior, the fractional order is modeled as a bounded, smooth function of an internal state variable $s(t)$:

$$q(t) = q_0 + \mu \cdot \tanh(\lambda \cdot s(t)), \quad (5)$$

Where:

- $q_0 \in (0,1)$ is the baseline memory level,
- $\mu > 0$ controls memory variability,
- $\lambda > 0$ defines sensitivity to the chosen state variable $s(t) \in \{x(t), y(t)\}$.

This formulation ensures realistic adaptation (deeper memory in confident phases), a bounded range $q(t) \in (q_0 - \mu,\ q_0 + \mu) \subset (0,\ 1)$, and smooth dynamics due to the hyperbolic tangent.

3.1 Selection of State Variable:

Two state-dependent configurations are considered in this study:

1. Investment-driven memory: $q(t)$ depends on $y(t)$, reflecting market sentiment and capital allocation. Figure 1 shows how investment demand influences the memory order. The curve exhibits bounded nonlinear behavior, consistent with theoretical expectations and observed financial dynamics [10].

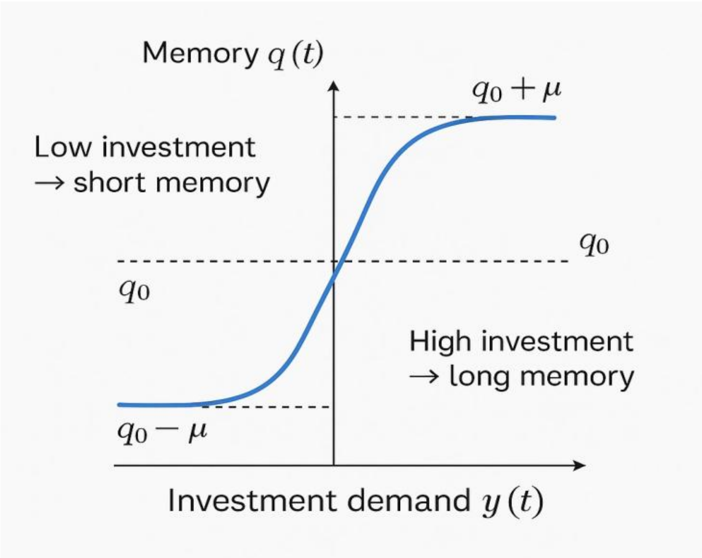

Figure (1): Adaptive memory order as a function of investment demand $y(t)$.

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2. Interest-driven memory: $q(t)$ depends on $x(t)$, representing monetary control and policy response. Figure 2 illustrates how interest rate changes affect the memory order. Similar to the investment-driven case, the curve is bounded and nonlinear, aligning with financial intuition and empirical behavior [10].

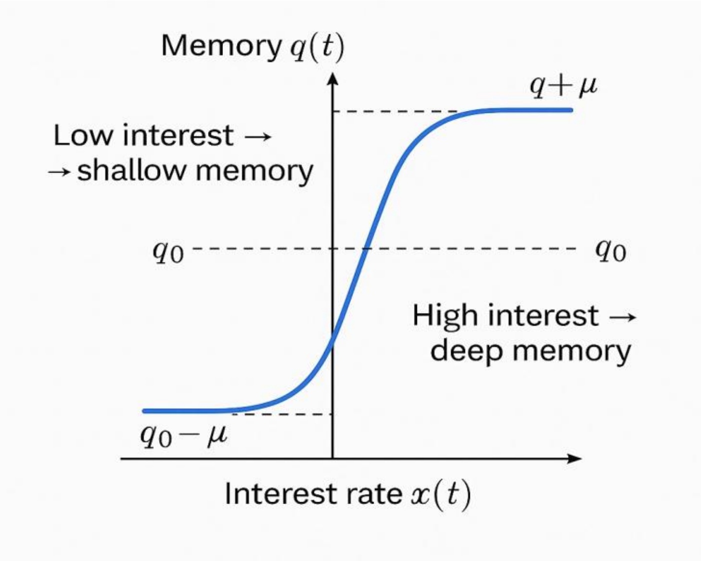

Figure (2): Adaptive memory order as a function of interest rate $x(t)$

3.2 Numerical Simulation:

We simulate the adaptive-memory model using the Caputo–Fabrizio derivative under two state-dependent scenarios. In both cases, the governing equations are the same as system (4), solved with a modified Euler method for variable-order dynamics [8, 3]. The step size is set to $h = 0.01$ and the total simulation time to $T = 100$. To ensure comparability between the two models, both formulations use the same set of initial conditions and parameter values, following prior studies such as [9]. This enables isolated analysis of the impact of the chosen memory-driving variable on the system dynamics.

3.2.1 Investment-Driven Memory:

When the memory order depends on investment demand y (t), the variables exhibit damped oscillations with sharper initial transitions. The adaptive memory shortens during demand drops and lengthens as demand stabilizes, as shown in Figure 3.

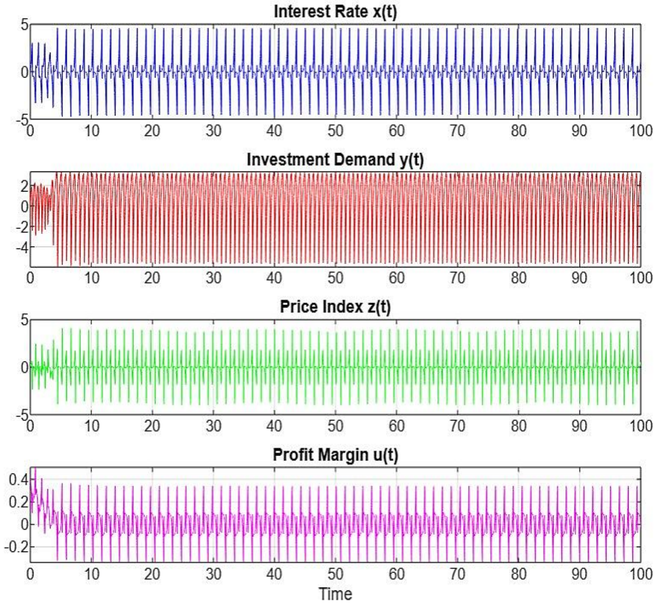

Figure (3): State trajectories under investment-driven memory

3.2.2 Interest-Driven Memory:

When the memory order depends on the interest rate x (t), the system displays smoother oscillations with gradual damping. Since interest rates typically vary more steadily than investment demand, memory changes more gradually, resulting in more predictable dynamics Figure 4.

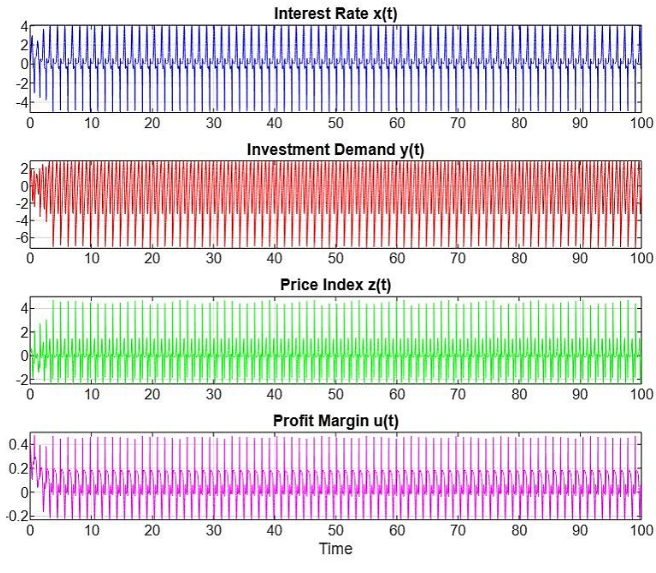

Figure (4): State trajectories under interest-driven memory

3.3 Stability and Economic Interpretation:

The stability of financial systems is closely linked to their economic meaning, as mathematical dynamics often reflect underlying market mechanisms. By examining system behavior under different memory structures, we can better understand how adaptive features influence both long-term stability and financial interpretation.

Stability and Long-Term Dynamics. Both formulations converge to bounded steady states, but with distinct transient behaviors. When memory adapts to investment demand $y(t)$, the system exhibits stronger initial oscillations, reflecting the reactive and volatile nature of investor sentiment. In contrast, when memory adapts to the interest rate $x(t)$, the convergence is more gradual and symmetric, consistent with the steadier adjustment of monetary policies.

Economic Interpretation. These differences highlight two complementary perspectives of financial behavior. A memory structure linked to $y(t)$ emphasizes market-driven volatility and short-term fluctuations, while memory linked to $x(t)$ underscores the stabilizing role of policy-driven dynamics. Together, they illustrate how adaptive memory can capture both instability and resilience in financial systems.

Table (1): Comparison between Investment-Based and Interest-Based Memory Models

Aspect	**Investment-Driven Memory ($q(y(t))$)**	**Interest-Driven Memory ($q(x(t))$)**
Driving Variable	Investment demand $y(t)$	Interest rate $x(t)$
Source of Dynamics	Reactive to market sentiment and demand	Guided by monetary policy and regulations
System Reactivity	Higher short-term volatility and responsiveness	Greater long-term stability and inertia
Modeling Focus	Speculative behavior, investor psychology, market-driven shocks	Monetary policy impact, systemic liquidity cycles
Numerical Behavior	Faster initial oscillations, then decay	Symmetric, damped responses
Recommended Application	Behavioral finance, high-frequency markets	Macroeconomic modeling, policy evaluation

3.4 Empirical Validation Using Real Financial Crisis Data:

To evaluate the applicability of the proposed models, we validate them against real-world data from the 2008 global financial crisis a period marked by investment collapse and aggressive monetary interventions.

Selected Variables and Data Sources. Two macroeconomic indicators are used as proxies for the model's state variables:

- Interest Rate $x(t)$: Effective Federal Funds Rate from the Federal Reserve Economic Data (FRED) [13, 14].
- Investment Demand $y(t)$: Gross Private Domestic Investment (GPDI) as a percentage of GDP from the U.S. Bureau of Economic Analysis [15, 16].

The series cover 2006–2012, spanning pre-crisis, crisis, and recovery phases.

Observed Patterns and Model Alignment. The Federal Funds Rate dropped from above 5% in 2007 to nearly zero by 2009, consistent with our interest-driven model predicting a fall in $q(t)$ and higher reactivity [13, 14]. Investment demand fell by more than 25% during 2008–2009, producing low $q(t)$ in the investment driven model, matching observed volatility [15, 16]. As recovery began post-2010, both indicators stabilized, leading to rising $q(t)$ and longer-term dynamics [8, 9].

Implications and Model Validity. The analysis confirms that memory contracts in crisis phases (low $x(t)$, $y(t)$) and expands in recovery, supporting long-term planning. This aligns with macro-financial evidence on adaptive expectations and endogenous memory [17, 18].

Potential for Data-Driven Calibration. The results suggest that estimating $q(t)$ directly from data is feasible. Methods such as recursive filtering, neural estimation, or adaptive Bayesian inference could be employed for real-time calibration [18, 5].

4. Challenges and Limitations

The proposed state-dependent variable-order financial model improves realism by linking memory to internal variables, but it also introduces notable challenges. First, the use of variable-order fractional derivatives increases mathematical and numerical complexity. Stability and convergence are harder to guarantee, especially when memory order $q(t)$ evolves dynamically [8, 7].

Second, the model is sensitive to its parameters q_0, μ, and λ, which control the range and responsiveness of memory. Without empirical calibration, these parameters may not reflect real financial behavior [10, 9]. Finally, while the model aligns qualitatively with crisis data, more rigorous validation using real-world multivariate datasets is needed to assess predictive value and robustness across economic conditions [18, 14].

5. Future Directions

The development of adaptive memory models opens multiple avenues for future research. One important direction is to calibrate the memory order $q(t)$ directly from empirical financial indicators—such as investment volatility, inflation expectations, or market risk indices—using data-driven methods like filtering or machine learning [18, 5]. This would improve model realism and allow real-time forecasting.

Another promising path involves extending the model to multi-agent or networked financial systems, where different sectors or institutions operate under heterogeneous memory profiles. Such extensions could reflect asymmetric responses to shocks or policy interventions. Finally, future studies could explore alternative functional forms for $q(t)$, including sigmoidal, piecewise, or adaptive rule-based structures, to test their impact on stability, responsiveness, and alignment with behavioral finance theories [7, 3, 17].

6. Conclusion

This work proposed a state-dependent fractional financial model using the Caputo-Fabrizio derivative with adaptive memory order $q(t)$. By linking memory to key internal variables—investment demand and interest rate—the model captured realistic economic behaviors such as short-term reactivity and long-term stability.

Simulation results confirmed that both memory mechanisms produce stable dynamics, each reflecting different aspects of financial behavior. The findings suggest that adaptive memory improves the flexibility and realism of fractional models, offering a promising approach for analyzing and forecasting complex financial systems.

Author Contributions:

Formal analysis: H.M. and J.F.A.; Investigation: H.M. and J.F.A.; Writing—original draft preparation: H.M. and J.F.A.; Writing—review and editing: H.M. and J.F.A. All authors have read and approved the final version of the manuscript and agree to be accountable for all aspects of the work.

Data Availability Statement:

The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

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