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The Differential Complex Equation $Pu = f$: A Solution Using the Kernel Method

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Abstract

In this study, a general approach to solving partial differential equations in what are called "differential complexes." The method is based on the main principle of breaking the equation's basic answer into two parts. This decomposition makes it possible to establish the criteria required for the existence of solutions as well as to create explicit formulas for solutions, making it a potent and adaptable tool that goes beyond the constraints of conventional approaches.

Keywords: Cohomology, Partial Differential Equation, Differential Complex.

1. Introduction

The study of partial differential equations (PDEs) of the form Pu=f is one of the main areas of mathematical analysis. While traditional methods, such as the Cauchy-Fantappiè formula and other integral representation-based techniques [1,5], have shown great efficacy in some areas, such as complex analysis on strongly pseudo-convex domains; nonetheless, the algebraic nature of the underlying spaces often limits their use. In this work, a more broad kernel approach to solving Pu=f in the context of differential complexes ($\varepsilon^{\cdot}$) is investigated. especially homotopically comparable to elliptic complexes or elliptic. The key insight of the method is the breakdown of the fundamental solution, which allows the development of generalized homotopy formulas.

2. Differential Complexes and Fundamental Solutions

A differential complex is a sequence of modules and differential operators:

$$\boldsymbol{\varepsilon^0 \overset{P_0}{\rightarrow} \varepsilon^1 \overset{P_1}{\rightarrow} \varepsilon^2 \overset{P_2}{\rightarrow} \cdots}$$

Because $P_{i+1}P_i = 0$ is the defining property, any two successive operators have zero composition. The Dolbeault complex in complex analysis and the de Rham complex in differential geometry are two concepts that are generalized by this structure.

A basic solution θ for the complex at degree q as a "generalized inverse" of the operators P_i, is a collection of kernels satisfying a system of equations. For example, the solution to $Pu = f$ is sometimes expressed as $u = \int \theta(x,y)f(y)dy$ an integral incorporating this fundamental answer.

3. Decomposition of the Fundamental Solution

The central achievement of this approach, as detailed in the following theorem, which is proved by [5].

Theorem 3.1:

Assume that Y is an open subset of X such that $X\backslash\bar{Y}$ is q-concave with respect to $E^{\cdot\prime}\ |_{X\backslash\bar{Y}}$. Then there exist operators $K_{q+1}^{(-1)} \in L(\varepsilon'^{q+1}(X\backslash\bar{Y}) \to \varepsilon^{q}(Y))$ and $K_{q+1}^{(-2)} \in L((\varepsilon'^{q+1}(X\backslash\bar{Y}) \to \varepsilon^{q}(Y))$ such that:

- $K_{q+1}^{(-1)}P_q = 0, and$
- $\theta_{q+1} = K_{q+1}^{(-1)} + K_{q+1}^{(-1)} + P_{q-1}K_{q+1}^{(-2)}\ on\ \varepsilon'^{q+1}(X\backslash\bar{Y}).$

The decomposition of the essential solution θ. The domain's q-concavity of the set X\Y is a critical topological and algebraic condition that permits this decomposition.

If a set's cohomology groups H disappear for any degree $i \le q$, it is said to be q-concave with respect to a complex. According to this condition, several associated differential equations can always be solved on that set.

Assuming this, the fundamental solution θ_{q+1} can be divided into two halves, $K_{q+1}^{(-1)}$ and $K_{q+1}^{(-2)}$ that meet the following requirements:

1. $K_{q+1}^{(-1)}P_q = 0$
2. $\theta_{q+1} = K_{q+1}^{(-1)} + P_{q-1}K_{q+1}^{(-2)}$

This decomposition is important because it distinguishes between the portion of the fundamental solution that is in the image of the prior operator and the portion that "annihilates" the operator P_q. The secret to creating clear solution formulae is this separation.

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4. Applications and Solution Formulas

The decomposition of the fundamental solution directly leads to powerful results for solving $Pu = f$:

4.1 Homotopy Formulas:

Theorem 4.1:

The following formula holds for each section $f \in C^{p_q-1}_{compp}(E^q\mid_{\bar{U}})$ satisfies $Pf = 0$ in U if U has a piecewise smooth boundary and is a neighborhood of Y:

$$f(x) = -K^{(-1)}_{q+1}([\partial U]^P f) + P(-K^{(-2)}_{q+1}([\partial U]^P f)) + \theta(\chi U f))(x \in Y)$$

Proof. In fact, using Corollary 2.4.9[5], we obtain

$$f(x) = -\theta_{q+1}([\partial U]^P f) + P\theta(\chi U f)(x \in U) \quad 4.1$$

Since $[\partial U]^P f \in E^{;'}_{q+1}{}'(X\backslash\bar{Y})$, We may apply Theorem 3.1's decomposition to the integralθ_{q+1} $([\partial U]^P f)\ for\ x \in Y$. This gives precisely formula (4.1). ■

The value of f inside Y can be represented in terms of integrals over the boundary ∂U for a section f satisfying $Pf = 0$ in a neighborhood U of Y with a piecewise smooth border. The conventional Cauchy-Fantappiè formula is powerfully extended to arbitrary differential complexes in this way.

$$f(x) = -K^{(-1)}_{q+1}([\partial U]^P f) + P(-K^{(-2)}_{q+1}([\partial U]^P f)) + \theta(\chi U f))(x \in Y)$$

4.2 Solvability Conditions:

Lemma 4.2:

Let U be a neighborhood of Y with a piecewise smooth boundary. In order that for a section $f \in C^{p_q-1}_{compp}(E^q\mid_{\bar{U}})$ there exists a section $u \in C^{p_q-1+p_q-2}_{compp}(E^{q-1}\mid_{\bar{U}})$ satisfying $Pu = f$ in U, it is necessary that

$$\int_{\partial u} GP_q(g; f) = 0\ for\ all\ g \in Z_{q+1}(E(\varepsilon^{\cdot\prime}\mid X\backslash\bar{Y})). \quad 4.2$$

Proof:

Actually, by applying Stokes' Theorem, we obtain

$$\int_{\partial u} GP_q(g;f) = \int_{\partial u} GP_q(g;Pu) = -\int_{\partial u} GP_{q-1}(P'g;u) = 0$$

as desired. ■

Corollary 4.3:

Assume that $f \in C^{p_q-1}_{compp}(E^q \mid_{\bar{U}})$ meets both condition (4.2) and $Pf = 0$ in U, It has piecewise smooth bounds and is a neighborhood of $\bar{Y}$. In Y, $f = Ph(f)$, where $h(f) = -K^{(2)}_{q+1}([\partial U]^P f) + \theta(\chi U f)$.

Proof. This follows immediately from Theorem 4.1 because

$$K^{(-1)}_{q+1}([\partial U]^P f)(x) = \int\limits_{\partial U} G_{p_q}\left(K^{(-1)}_{q+1}(x,.), f\right) \quad (x \in Y),$$

And $P'(y)K^{(1)}_{q+1}(x;y) = 0$ for (x,y) in the product $Y \times (X \backslash \bar{Y})$ ■

This method offers a precise criterion for the existence of a solution. According to Lemma 4.2, a particular border integral must disappear in order for a solution to exist:

$$\int_{\partial u} G_{pq}(g;f) \qquad \forall\, g$$

Furthermore, corollary 4.3 shows that this required condition becomes sufficient when q-concavity is assumed. It also provides an explicit formula for the solution u:

$$u = h(f) = -K^{(2)}_{q+1}([\partial U]^P f) + \theta(\chi U f)$$

5. Conclusion

The suggested method expands the framework from conventional solutions to a more thorough one, marking a significant change in the way differential equations are solved. By establishing a strong connection between the algebraic characteristics of the equation and the topological characteristics of the space on which it is solved, the decomposition of the fundamental solution offers a potent tool that not only identifies solutions but also establishes the necessary and sufficient conditions for their existence. The study further highlights the essential connection between the geometric structure of the domain and the solvability of equations by demonstrating that topological criteria like q-concavity are essential to guaranteeing the solvability of equations.

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