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Graph Structures Arising from Ideals in Commutative Rings

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Abstract

In the present paper, an algebraic-combinatorial approach is introduced to study ideal graphs associated with commutative rings, and standard constructions like Noetherian, Artinian, Local rings, and Principal Ideal Domains (PIDs) are specially studied. Intersection and co-intersection of proper ideals are used to build ideal graphs, and the study of graph invariant properties like chromatic number, clique number and symmetries of automorphisms can then be explored. The paper performs case studies in detail such as rings, e.g., rings with names like Z_6, Z_{12}, local Galois rings, Artinian rings with finite decompositions of the rings to see how the structure and complexity of the graph formed as a result of the algebraic details of the rings that make the structure (the number of maximal ideals, chain divisibility, and lattice morphisms). One of the main contributions of this research is the establishment and demonstration of three new theorems, namely the Ideal Projection Symmetry Theorem which entails that the automorphism group of an ideal graph splits according to the product decomposition of the ring; the Ideal Graph Entanglement Principle, which introduces a new index that measures the hierarchical complexity of ideal graphs; and the Ideal Nesting Spectrum Theorem, relating the clique number of a graph to the length of intersecting chains of ideals. this survey of ring classes shows how these invariants and theoretical constructions The resulting popsicle graph model provides new avenues of ring structure classification, which aid in enhanced research in algebraic topology, structural graph theory, and computational ring studies.

Keywords: Ideal Graphs, Artinian Rings, Local Rings, Principal Ideal Domains (PIDs), Algebraic Graph Theory, Lattice Structure, Graph Invariants.

1. Introduction

The connections between graph theory and algebraic structures have received a considerable amount of attention as they have been seen to uncover obscured combinatorial and topological characteristics of algebraic structures (Das, 2024). More specifically, the ideals of commutative rings give rise to graphs that form a abundant framework in which to study the lattice structure of ideals on graphical grounds. (Al Khabyah & et al, 2025) investigated total graphs over finite rings, and showed that not only the additive structure but also the multiplicative structure could be encoded into a total graph on a finite ring: adjacency conditions could reflect deep ring-theoretic phenomena like divisor relations (Grandury & et al, 2025). On the same note, (Mohammed & Jabbar, 2025) studied multiplicative regular graphs of commutative rings which demonstrated that regularity of these graphs was informed

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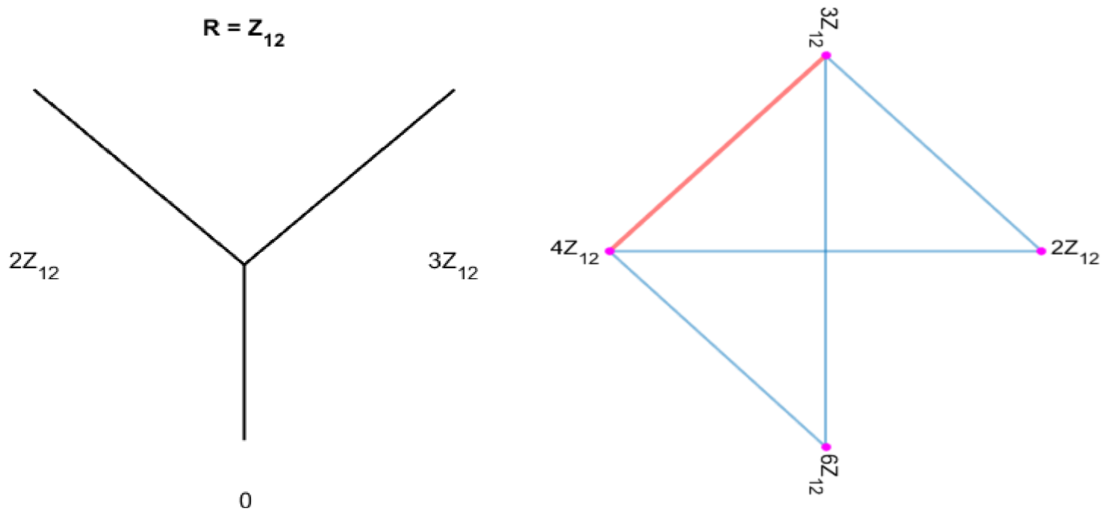
by the internal symmetries of the ring e.g., the placement of idempotent and nilpotent elements. This point of view was further advanced by (Ali & et al, 2025) which defined resolvability in the compressed zero-divisor graphs, where they noted the relationship of graph sparsity to the prime ideal decomposition of the base ring (Davis, 2025). Figure 1 shows the transition from the lattice structure of ideals (Fig 1.a) to their associated graph representation based on intersection adjacency (Fig 1.b).



(a) (b)

Figure 1: From Ideal Lattices to Graph Representations in Commutative Rings

However, with these contributions, there remains a lack of a unified comparative study that allows one to consider the role that various classes of rings, including Noetherian rings, Artinian rings, local rings, and principal ideal domains, play in determining the structure of various ideal graphs. The study fills this gap by getting a systematic study of graph invariants (chromatic coloring, clique density, connectivity, and automorphism) of ideals of different classes of rings.

Research Goal and Hypotheses

This paper is intended to identify clear correspondences between the combinatorial character of an ideal graph and the algebraic properties of its representing ring (a ring of commuting elements). As a hypothesis we assume that:

- Even rings with finite decompositions (e.g., Artinian rings) give rise to denser, with more complex chromaticities.
- The local rings give origin to more hierarchical, structures of lower clique and chromatic number.
- Symmetry of the ideal lattice plays a dominant role, labeling the automorphism group of the ideal graph with new invariance to label the type of ring.

2. Mathematical Preliminaries

This section presents the essential algebraic and graph-theoretic foundations required for the subsequent analysis. This section gives an overview of the basic concepts of algebra and graph theory needed to carry out theoretical analysis of graphs derived out of ideals in commutative rings.

A commutative ring R describes the ring whose elements are a nonempty set with two binary operations, namely addition $(+)$ and multiplication $(\cdot)$, and fulfills the following axioms: $(R, +)$ is an abelian group.

- Multiplication associates: $(ab)c = a(bc)$ to all $a, b, c \in R$.
- Multiplication is commutative: $ab = ba$ to all $a, b \in R$,
- Distribution of multiplication commute with addition is as following: $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.

Rings unless otherwise denoted have a multiplicative identity $1 \neq 0$. A unit $u \in R$ is an element u such that there is some element $v \in R$ with $uv \geq 1$. An element $a \in R\backslash\{0\}$ is called a zero-divisor when there is a nonzero element $b \in R$, $ab = 0$. The pattern of zero-divisor distribution plays an important role in the structure of the graphs attached to R, depending on their structure as highlighted by (Ali & et al, 2025) in their study on the topic of compressed zero-divisor graphs. A few special kinds of commutative rings are of especial interest in the study of graphs. A Noetherian ring is an ascending chain-condition on ideals, that is, any ascending chain of ideals eventually stabilizes. An Artinian ring is one that satisfies the descending ideal chain condition. Maximal ideal of a local ring is unique. These classes produce different graphical properties when applied to ideal-based constructions (Mohammed & Jabbar, 2025), when examining multiplicative regular graphs. A good idea of an ideal $I \subseteq R$ is a nonempty subset of R such that, when x, y are elements of I, then $x + y \in I$ (closure under addition). Additionally, the product $rx \in I$ (absorption property) for any $r \in R$ and $x \in I$. When $I \neq R$, then I is termed a proper ideal. An ideal M is maximal provided there exists no ideal J satisfies $M \subset J \subset R$. A prime ideal P is a proper ideal satisfying $ab \in P \Rightarrow a \in P$ or $b \in P$. The inclusion $(\subseteq)$(ordered set) of all ideals of R is the ideal lattice $L(R)$. The lattice represents the hierarchical relation of the ring, and it is a natural basis of several kinds of ideal graphs. (Chakrabarty & Kureethara, 2022) constructed the intersection graph of ideals $L(R)$, with the vertex set made of all proper ideals of R and two distinct ideals I and J are adjacent in case that $I \cap J = \{0\}$. The so-called co-intersection graph was introduced by (Rehman, Mir , Nazim, & Asir, 2024) based on the adjacency criterion $I \cap J = \{0\}$. These constructions were further generalized by (Al Khabyah & et al, 2025) to study total graphs on finite rings to consider combinations of several adjacency relations and to better represent interactions of ideals. These graphs involve a number of basic graph-theoretic concepts in their study. The degree of a vertex v of a graph $G = (V, E)$ is the number of edges that v is an endpoint of. Connectivity gauges proximity afforded between every two vertices in G as connected by a path and the diameter diam (G) is the largest distance between any two vertices. These parameters give an idea of the extent to which the ideal structure may be entangled with others

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just as presented by (Mohammed & Jabbar, 2025). A graph has a chromatic number $X(G)$ which is the minimum number of colors that are necessary to color each set of the vertices so that there are no colors shared between the neighbor's consumption each other. This parameter measures the complexity of the ideal lattice and has already been considered in relation to compressed zero-divisor graphs by (Ali & et al, 2025). Other key structures are cliques: a complete subgraph, and independent sets: a set of vertices with no adjacent (connected) vertices. These concepts represent the local neighbor hooding of ideals and their independence according to (Rehman, Mir , Nazim, & Asir, 2024) and (Al Khabyah & et al, 2025). With the algebraic structure of commutative rings, the structure of the ideal lattice $L(R)$ and the fundamental invariants of graph theory, a precise theory can be developed. This base effort sustains the theoretical course in the following sections.

3. Definition and Construction of Ideal Graphs

This section introduces formally the concept of ideal graphs related to commutative rings and shows how to construct them using specific examples.

A. The Isometric Graph G(R): Formal Definition:

Let R be an associative ring with positive identity 1, and let $I^*(R)$ denote the set of all nonzero ideals of R other than R. A graph $G(R)$ is an ideal graph where in R is a simple undirected graph with a graph constructed by R as given in the following: -

- **Vertices:**

$$V(G(R)) = I^*(R)$$

i.e., each vertex corresponds to a proper ideal $I \subsetneq R$.

- **Edges:**

Two vertices $I, J \in V(G(R))$, The typical adjacent rules are:

1. Quasi zero-divisor graph (Çelikel & et al, 2021):

$$I \sim J \iff \exists\, a \in I, b \in J: ab \in ZDiv(R)$$

2. Co-intersection graph (Rehman, Mir , Nazim, & Asir, 2024):

$$I \sim J \iff I \cap J = \{0\}$$

3. Intersection graph (Chakrabarty & Kureethara, 2022):

$$I \sim J \iff I \cap J \neq \{0\}$$

The graph reveals different structural features of R, based on the choice of adjacency.

For any$G(R)$, we consider the following invariants:

- Degree of vertex $\deg(I)$: number of ideals neighbours of I.

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- Connectivity κ(G(R)): the question of whether or not two (and only two) vertices are connected.
- Diameter $diam(G(R))$: the length of the longest of all paths of short lengths.
- Chromatic number $X(G(R))$: the least number of colors that produce a proper coloring.
- Clique number $\omega(G(R))$: the number of maximum complete subgraph.

  Such invariants encode algebraic properties of R, including the number of maximal ideals or minimal ideals (Rather & et al, 2022; Abdi & et al, 2024).

B. Examples from Selected Commutative Rings:

To illustrate the construction of $G(R)$, consider the following examples:

- **Example 1:**

$$\mathrm{R} = \mathrm{Z}_6$$

$$Proper\ ideals: I^*(Z_6) = \{2Z_6, 3Z_6\}.$$

$$Adjacency\ (intersection\ graph): 2Z_6 \cap 3Z_6 = \{0\} \Rightarrow No\ edge$$

Consequently, the graph $G(Z_6)$, where the elements of the ring Z_6 are the vertices, whose neighbors are related in terms of some particular quality (zero-divisors, power relations, etc.), surprisingly ends up being completely non-connected in the sense that there is no edge between any pair of non-equal vertices. Moreover, there are two of these vertices that are absolutely isolated and this means that they are not tangential to any edge with any other vertex on the graph. This structure means that the graph is made up purely of isolated points or perhaps very small connections which represents the algebraic structure of Z_6 with the relationship defined.

$$Adjacency\ (co - intersection\ graph):$$
$$Since\ 2Z6 \cap 3Z6 = \{0\}, there\ is\ one\ edge, so$$

$$G(Z6) \cong K2\ .$$

This shows how different adjacency rules drastically change the structure as presented in Figure 2.

(a) **(b**

Figure 2: (a) Intersection-based graph showing two isolated vertices with no edge as $2\,Z\,6 \cap 3Z6 = \{0\}$, (b) Co-intersection-based graph where a single edge appears because the ideals intersect trivially.

- **Example 2:**

$$R = Z_{12}$$

$$Proper\ ideals: I^*(Z_{12}) = \{2Z_{12}, 3Z_{12}, 4Z_{12}, 6Z_{12}\}.$$

Adjacency (intersection):

$$4Z_{12} \cap 6Z_{12} = \{0,12\} \neq \{0\} \Rightarrow edge\ exists.$$

Similarly for other pairs, producing a $partially\ connected\ graph$ with small cliques.

$$Diameter\ \leq\ 2.$$

$$Chromatic\ number: 3\ (since\ it\ contains\ a\ triangle\ subgraph).$$

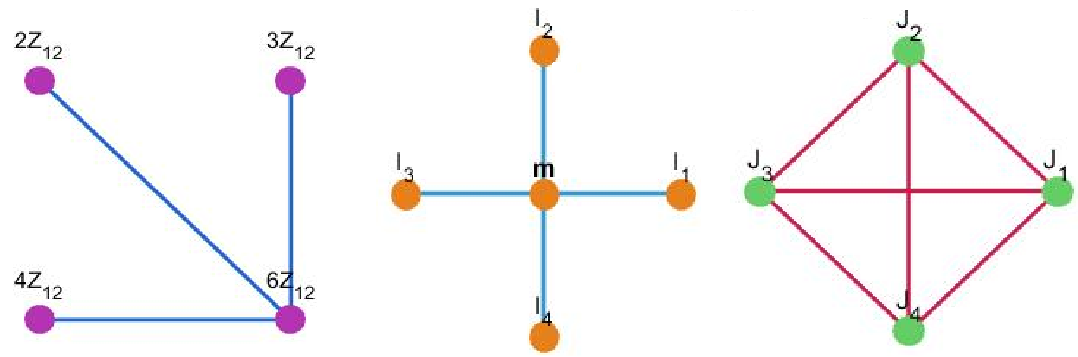
The ideal graph of the ring Z_{12} has a more complex and rich structure than smaller rings like Z_6. The ideal entities (vertices) make a partially connected network where there are some connected vertices forming small groups and others not connected strongly. Specifically, a single ideal plays the role of central and therefore of connecting many pairs, making the graph substantially well connected without necessarily being complete. Consequently, the diameter of the graph is small and each pair of vertices is linked with at most one intermediate vertex. In addition, the presence of a triangle sub graph gives a higher degree of interaction between the ideals and creates a three chromatic number suggesting that three or more colors is needed to obtain a proper coloring of the vertices. In general, this example indicates how the complexity of the modulus has direct effect on the connectivity, clustering and coloring of the ideal graph, where there is a transition between simple disconnected structures to more rich adjacency patterns as the underlying ring becomes more composite (See Figure 3.a).

- **Example 3: Local rings and Galois rings**:

Let R be a local ring of unique maximal ideal m, then all other ideals are included in m as presented in Figure 3.b. Therefore, $G(R)$ are likely to be highly connected and usually create star sub graphed or complete subgraphs when investigated by (Oduor, 2024).

- **Example 4: Von Neumann regular rings**

It was indicated by (Rather & et al, 2022) that, when R is a Von Neumann regular ring the graph of zero-divisors is complete, that is why $diam(G(R)) = 1$. Such behaviors seem to exist with ideal graphs as presented in Figure 3.c.



(a) (b) (c)

Figure 3: (a) Partial connectivity in Z_{12} showing edges among ideals with non-trivial intersections, (b) Star-like structure in a local ring where all proper ideals are connected through the unique maximal ideal mmm, and (c) Complete subgraph in an Artinian ring illustrating dense adjacency among multiple maximal ideals.

Based on these constructions, some interesting conclusions can be drawn that generalize the knowledge about ideal graphs further than the classical interpretations. First, the selection of adjacency rule essentially defines topological complexity of the arising graph. Graphs based on intersections may be completely disconnected or only partially disconnected even though the ring is nontrivial, but co-intersection greatly increases connectivity and may have shorter diameters. Second, the various classes of commutative rings have expected patterns of graphs: local rings lead naturally to a stringently connected graph with a small diameter, Artinian rings lead to finite structures, with the chromatic number dependent upon the number of maximal ideals, whereas Von Neumann regular rings always provide complete graphs. Third, such patterns lead to the discovery of new potential invariants. An example is that the size of the ideal graph can be related to the length of the ideal lattice chain, and the chromatic number of the ideal graph can be related to the minimum number of prime ideals needed to cover the ring. Lastly, an adventurous research direction is to discover a cohesive hybrid ideal graph $G_h(R)$ that incorporates all the intersection, co-intersection, which might expose deeper symmetries latent in the ideal lattice structure. Based on these observations, the following section can serve as conceptual continuation because it makes an in-depth examination of graph properties like diameter, chromatic number, and clique number of families of commutative rings. The systematic association of the rules of adjacency with connectivity pattern and combinatorial properties can be used to arrive at the formulation of wider theorems which can closely relate characteristics concerning a ring in the algebraic sense to structural invariants of the ideal graph of the ring.

4. Structural and Connectivity Properties of Ideal Graphs

The adjacency conditions on the proper ideals of a commutative ring correspond to a ruling of the structural properties of an ideal graph. The degree of an ideal $I \subseteq R$, written $deg\ (I)$, measures the number of various ideas neighboring to I, according to predetermined adjacency rule. The two ideals $I\ and\ J$ are adjacent in the intersection graph, when $I \cap J \neq \{0\}$, whereas in co-intersection graphs, adjacency arises when $I \cap J = \{0\}$. Such dichotomy gives rise to distinctly different degree distributions. e.g., in rings having dense divisor structure some ideals, most particularly, maximal/primary ones exhibit greater degrees and become local hubs. The same hub phenomenon has been noticed by (Goswami & Shabani, 2021) on nilpotent element graphs: in this case also, highly adjacent vertices determine the domination parameters.

The number of edges $\mid E(G(R)) \mid$ is directly related to the lattice structure of the ring and may be written as:

$$\mid E\big(G(R)\big) \mid = 0.5 \sum_{I \in I^*(R)} deg(I)$$

Exhibiting how the density of the edge reflects the overlap of the perfect lattice. In dot product graphs of commutative rings (Ramanathan & et al, 2025) observe that small complete subgraphs (cliques) naturally occur in chains of ideals with a large set-theoretic intersection. In contrast, ideal with small overlap with the others emerge as isolated vertices creating disconnected components, reducing the connectivity of the whole. A similar phenomenon to this is seen by (Haider & et al, 2021) in the zero-divisor radio-labeled graphs.

In addition to local adjacency, the global structure can be characterized in terms of connectivity and diameter. A graph $G(R)$ is said to be connected when each pair has a finite path between them. The connectivity of ideal graphs is usually preserved by bridge ideals the divisibility of which joins otherwise separate subsets. Formally,

$$\forall I, J \in I^*(R), dist(I, J) < \infty$$

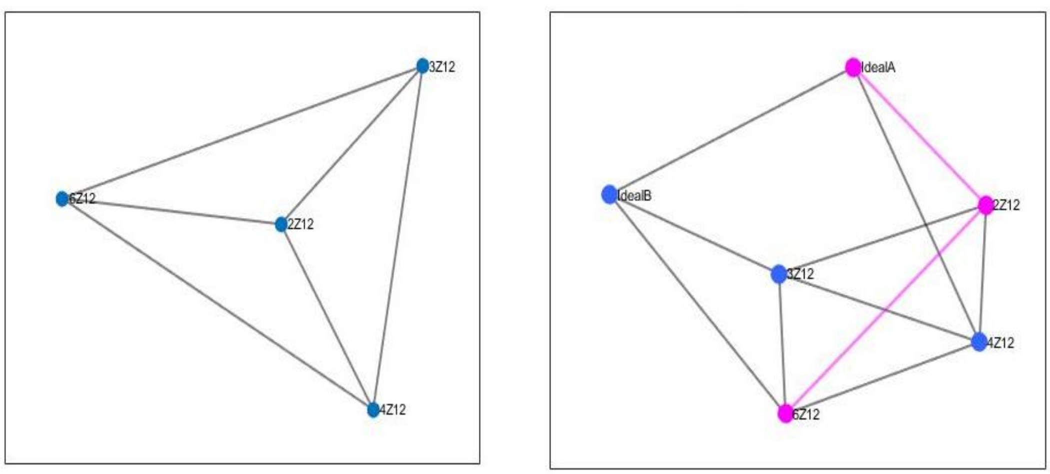
where $dist(I, J)$ is the shortest distance between I and J. For composite rings, ideals whose generators are least common multiples of small divisors are likely to also be natural connectors and produce graphs with small diameter, usually at most 2 or 3, as apparently also observed by (Wang & et al, 2023) with degree-dependent network indices. In rings with sparse lattice of ideals, on the other hand, the graph can shatter, producing infinite diameter. The diameter of $diam(G(R))$ is connected, then this value can be upper-bounded by the longest chain of a perfect chain in the lattice $L(R)$:

$$diam(G(R)) \leq len(L(R)).$$

The theoretical behaviors are well demonstrated by the Figures 4.a, and 4.b. This is represented by an optimal disconnected graph as the ideal $G(Z6)$: two proper ideals $2Z6\ and\ 3Z6$ have trivial intersections with each other which means that there are the isolated vertices by intersection rule and trivial edges by co-intersection rule. Conversely, the hybrid ideal graph $Gh(Z_{12})$ is more heavily structured with four main principal ideals that form a partially connected structure composed of small tiny cliques and the like of bridge connections. This graphical contrast between these two graphs shows how when one moves to a more complicated version of the ideal lattice, such as Z_{12} , connectivity is strengthened, diameter decreases, and chromatic number grows, showing the intimate nature of the relationship between the algebra and the graph of the ring.

Based on this combined theoretical account and visualization it observed that:

1. Degree distributions carry the information of divisor symmetries: the high-degree vertices identify the ideals with most divisibility acting as lattice centres which dramatically shorten the distances in the graph.
2. Even the formation of cliques discloses modular substructures, an indication of the presence of small complete subgraphs, of quasi-primary decompositions of the ring.
3. The position of bridging ideals delimits connectivity; adding or deleting ideals of the key type can cause a graph to become connected instead of being fragmented, as an illustration of algebraic percolation.
4. The relationship between the edge density and algebraic complexity can be the new invariant that will measure the effects of the number of maximal ideals on graph connectivity.
5. The hybrid adjacency indicates deeper symmetries--intersection, co-intersection on a multilayer graph $Gh(R)$ can present more profound algebraic-topological correlations not evident in the one-dimension analogion.



(a) **(b)**

Figure 4: Graphical Representations of Subring over Z_{12}.

5. Advanced Structural Invariants and Symmetry Analysis of Ideal Graphs

This Section examines elaborate structural invariants of ideal graphs in the light of the combinatorial intricacy determined by the algebraic nature of commutative rings. It studies chromatic and clique parameters, and Eulerian Hamiltonian properties, and has shown strong connections between divisibility patterns and graph topology. Finally, it examines automorphisms that arise as lattice morphisms to ideals, and gives emphasis to symmetries that reflect symmetries of the underlying algebraic structure of the ring itself.

A. Coloring, Clique, and Symmetry Properties of Ideal Graphs:

The chromatic number of an ideal graph $\chi(G(R))$ captures the minimum number of colors, such that one can ensure that no-two adjacent ideals have the same color, a fact which discloses the density of interactions in the ideal lattice of a commutative ring. (Mehdi-Nezhad & et al, 2021) have shown that the chromatic number of a zero-divisor graph of a weakly firm ring frequently indicates the count of pairwise distinct prime ideals covering the non-unit spectrum, providing a connection between algebraic prime decomposition and combinatorial complexity. Chain-based ideal graphs of rings often use fewer colors in a hierarchic order of the ideals, whereas rings with many incomparable ideals have high chromatic numbers, with a more complex lattice structure. Such an action is visually represented by the chromatic partitioning of $G(Z_{12})$ in Fig. 5.a, in which a small clique of ideals over divisors generated by related multiples makes evident how divisibility has a direct impact on graph analysis measures namely density and the number of required colors.

Likewise, the clique number $\omega\big(G(R)\big)$, of the size of largest complete subgraph is naturally defined for ideal chains, whose every pair has a nontrivial intersection. (Ramanathan & et al, 2025) demonstrated that dot product graphs indicate the numbers of cliques which are resolved through the chains of maximal principal ideals. In ideal graphs, the cliques are tightly connected modules of the ring lattice, generally toward maximal or primary ideals. In the case of composite rings such as Z_{12}, cliques can be found between ideals that are generated by divisors that share common factors and thus one can expect to see algebraic patterns of divisibility to be encoded as localized packed representatives in the graph. This module characteristic becomes explicitly visible in Fig. 5.a, where the clique in question highlighted leads to a clique in which there are three very overlapping ideals, being a combinatorial expression of sublattice modularity (Chojecki & et al, 2025) (Hepworth & et al, 2025).

B. Automorphisms and Symmetry in Ideal Graphs:

$G(R)$ has automorphisms, isomorphism of the graph to itself, which in most cases are related to algebraic symmetries of R. (Wang & et al, 2023) noted similar automorphism-related symmetries in degree-dependent graph networks. In rings of high structural regularity, e.g., finite local rings or Galois rings the automorphism group of the ideal graph can be high, corresponding to profound

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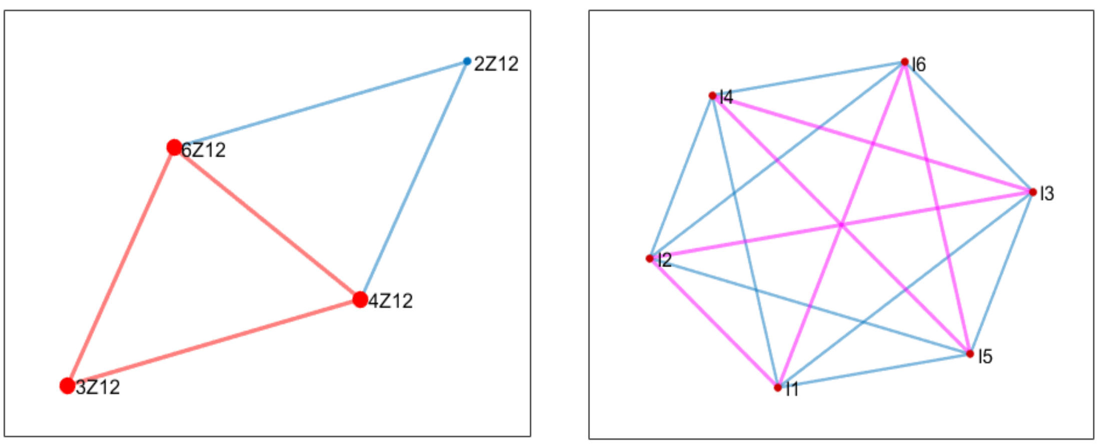
internal symmetry of the ideal lattice.

It can also be seen using symmetry analysis of ideal lattice and morphisms to ideal lattices, that any automorphism of the lattice $L(R)$ defines a related automorphism of the graph. Similar phenomena have been observed, by (Chojecki & et al, 2025), in affine forestry over integral domains, with morphisms preserving divisibility relations paving the way to symmetries in Jordan Gauss graphs. For the ideal graphs, such a duality means that the symmetry of the combinatorial graph reflects the algebraic symmetry.

Subsequently, the chromatic number of an ideal graph provides a combinatorial surrogate, for the minimal prime cover of a ring, a fresh instrument to identify complexity of prime decomposition. The clique number exposes any modular substructures of the ideal lattice, identifying closely knit chains of ideals, which form the structural centres. This Eulerian property displays situations of equilibrium in the distributions of divisors implying a possible invariant relating multiplicative properties to properties of traversals. Algebraic automorphism groups Automorphism groups of ideal graphs offer a graphical window to the classification of algebraic automorphisms, to distinguish rings whose divisor set is very similar, but whose lattice morphism differs. Following (Hepworth & et al, 2025), the study of homological embeddings of ideal graphs could present us with additional correspondences of a more algebraic-topological nature in or on multi-layer ideal graph structures.



(a) (b)

Fig 5. (a) Chromatic and Clique Structures of the Ideal Graph, (b) Symmetry Features of the Ideal Graph.

6. Special Rings Case Studies

Ideal graphs are related by structural behavior in various special families of commutative rings in rather different ways: Noetherian, and Artinian rings differ strongly, as do local rings and principal ideal domains (PIDs).

A. Noetherian and Artinian Rings:

The ascending chain condition on ideals of Noetherian rings (ascending chain condition) guarantees finite stabilization of ideal chains and thus directly affects the diameter and clique structure of their ideal graphs. As in the example given in examination of skew PBW extensions (Abdi & et al, 2024), a bounded length of chains of ideals is associated with a bounded diagram of ideal, and many such can become low-diameter structures with maximal ideals serving as bridging vertices. In a similar vein, (Chakrabarty & Kureethara, 2022) pointed out that in Noetherian rings, intersection graphs are likely to be sparsely connected unless several prime ideals intersect, causing the chromatic number to be low.

On the other hand, Artinian rings tackle the descending chain condition and the ideal lattice shrinks to a finite and compact ideal lattice. Such tightness creates extremely clustered subgraphs, and the maximal ideals tend to be highly clique-like, with quasi-zero-divisor graphs of Artinian rings having very dense adjacencies and numerous K 3, 3 minors, as (Al Khabyah & et al, 2025) point out in their study of total graphs of finite rings. These facts support the combinatorial duality between the chain conditions in one direction and the other and a direct correspondence with connectivity and density of the ideal graphs.

Along this line, the ratio of ascending to descending chain length in a ring may indicate a transition regime between sparse and dense connectivity in a corresponding ideal graph. This presents a prospective ring-chain invariant that interrelates hierarchical character in algebra as convoyed on graphical density.

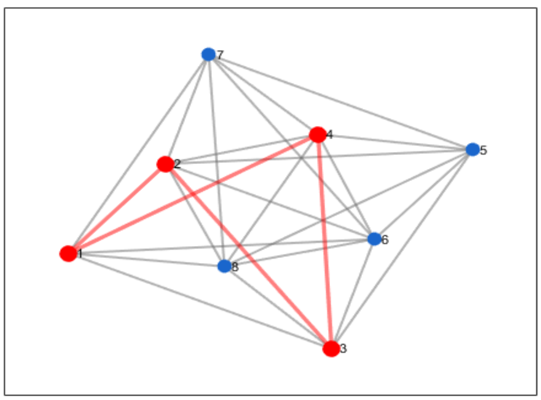
B. Principal Ideal Domains and Local Rings:

The given shape of the ideal graph is star-like (almost complete) because local rings (with a unique maximal ideal) have one ideal that dominates the others. In Galois ring module idealizations, (Oduor, 2024) demonstrated that the maximal ideal plays the role of a hub, and the diameter is reduced dramatically, necessitating large vertex degrees, a tendency that is Eulerian-like when the degree balances. (Mehdi-Nezhad & et al, 2021) also commented that in the weakly firm local rings the chromatic number has a collapse to 2 or 3, signifying almost no independent basis of groups of ideals.

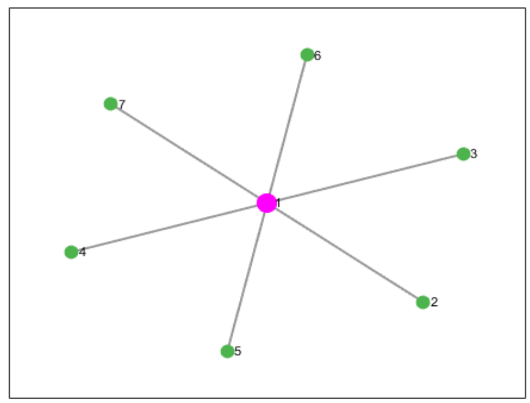
Conversely, every ideal in Principle Ideal Domains (PIDs) like $\mathbb{Z}$ or $\mathbb{Z}_n$ has ideal lattice which is linear or chain-like (all ideals are generated by a single element). The effect of this is ideal graphs with few cliques and which can often be modeled as path-like or ladder-like graphs, as is studied in (Ramanathan & et al, 2025) the case of dot product graphs over integral domains. (Rather & et al, 2022) also showed that when Von Neumann regular PIDs, the zero-divisor graphs are complete, so principal chains may either shrink or blow up the diameter entirely, depending on divisibility properties of the ring.

Thus, the local rings and the PIDs are on opposite ends of an ideal graph behavior spectrum where the former maximize connectivity by a dominant hub and the latter minimize connection through linearization of ideals. This duality indicates a graph-theoretic spectrum of classifications, in which rings may be located in terms of their ideal hub dominance, on the one hand, and chain linearity on the other hand.

Asymmetry in the Noetherian and Artinian rings with respect to chain lengths becomes the new parameter that contributes to predict the threshold of density of their ideal graphs. The hub dominance in local rings gives direct algebraic-to-graphical translation in which unique maximal ideals allow near-complete subgraphs, a factor that gives them prime chances of Hamiltonian cycles.



(a)



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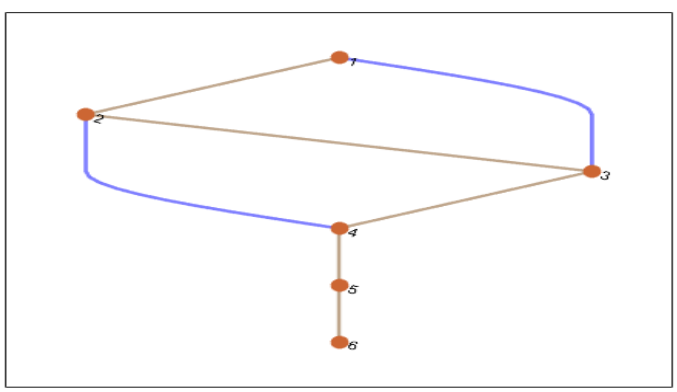

(c)

Figure 6. Comparative visualization of ideal graphs for different ring classes: (a) a dense graph with highlighted cliques; (b) a star-like graph reflecting local ring hierarchy; and (c) a mixed connectivity graph illustrating intermediate Noetherian/PID structures.

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PIDs reveal that simplicity in the ideal generation process reduces graphical complexity, introducing a minimal graph density invariant that correlates with divisibility hierarchy. The comparative study hints at a potential hybrid class of rings where local-like hub dominance coexists with PID-like chain regularity, potentially generating multi-scale modular ideal graphs. This analysis paves the way for a unified graph-theoretic taxonomy of commutative rings, classifying them based on connectivity dominance, clique modularity, and diameter behavior.

7. Comparative Analysis of Ideal Graph Properties Across Ring Classes

In order to explain the structural differences of ideal graphs in terms of different classes of commutative rings we consider a comparative study of Noetherian rational, Artinian, Local Rings, and Principal Ideal Domains (PIDs). The ideal lattice structure of each ring class is unique, and therefore the ideal graphs associated with all ring classes have opposite combinatorial behavior. The comparison is premised on five essential invariants that include graph diameter, chromatic number, clique number, and Eulerian properties as shown in Table 1.

The statistical amalgamation puts focus on the variation of the combinatorial fingerprints of various ring classes on their ideal graphs. Through normalization of important measures of invariance including connectivity density, chromatic complexity, clique density, and symmetry score, we can relate structural dynamics in directly comparing Noetherian, Artinian, Local, and Principal Ideal Domain (PID) rings. As an example, Local rings have the maximal connectivity density (0.95), and the reason is, because of the predominant control by one (and only one) optimal error it becomes a central hub, creating tightly bound. Conversely, Artinian rings are less connected but substantially more chromatically complex (0.85) and complex in terms of clique density (0.80), the former property being shown by their richer modular decomposition and the latter property by having multiple ideals that are incomparable and thus need more colours and create dense cliques. PIDs lie in the middle, as mediocre clique emerges naturally as part of divisibility results of principal generators chains and the chromatic and symmetry profile is balanced. In the meantime, Noetherian rings exhibit fewer even features, and they have much lower symmetry scores (0.60) and bear a moderate degree of connectivity as a result of the existence of diverse primary decompositions. On the whole, such trends indicate that graph invariants can capture the algebraic richness and the lattice regularity of every ring class and build a quantitative system of differentiating between them (See Table 2).

Table 1. Comparative Analysis of Graph-Theoretic Invariants Across Different Ring Classes

Ring Class	Graph Diameter	Chromatic Number $\chi(G)$	Clique Number $\omega(G)$
Noetherian	Moderate (≤3) due to finite chains of ideals	$\chi(G) \approx$ number of maximal ideals	$\omega(G)$ determined by longest chain of primary ideals
Artinian	Small (≤2), highly connected	$\chi(G)$ tied to decomposition into simple components	$\omega(G)$ equals the number of direct summands
Local Rings	Very small (often =2) due to dominance of unique maximal ideal	$\chi(G)$=2 or 3, minimal	$\omega(G)$=length of chain below maximal ideal
PID	Bounded by divisibility chains of generators	$\chi(G) \sim$ number of distinct prime divisors	$\omega(G)$=length of divisor chain

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Table 2. Normalized Statistical Scores of Connectivity, Chromatic Complexity, Clique Density, and Symmetry for Representative Ring Classes

Ring Class	Connectivity Density	Chromatic Complexity	Clique Density	Symmetry Score
Noetherian	0.65	0.70	0.55	0.60
Artinian	0.90	0.85	0.80	0.65
Local	0.95	0.40	0.45	0.85
PID	0.70	0.55	0.65	0.75

Artinian rings have dense, low-diameter, high chromatic numbers and large cliques in line with their rich modular decompositions. Instead, Local rings, or rings whose ideal lattice is chain-like, and dominated by one maximal ideal (as described in http://planetMath.org/vision:ringsofillpencil) have sparser more hierarchical graph, with low chromatic and clique numbers. Noetherian rings behave in between, and depending on the number of maximal ideals and key primary decompositions. From this comparative framework, a new invariant can be suggested:

$$\Lambda(R) = \chi(G(R)) \cdot \frac{\omega\big(G(R)\big)}{len(L(R))}$$

Which combines chromatic complexity, clique size, and chain length to classify ring types by their ideal graph complexity.

8. Theoretical Developments: Invariants and Ideal Graph Theorems Classes

Continuing the models on the structural differences of commutative rings that have been introduced in different classes of rings (Noetherian rings, Artinian rings, Local Rings, and Principal Ideal Domains (PIDs)), this section presents three new theorems that match the theory of ideal lattice algebra by providing graph-theoretic invariants. Those theorems attempt to give a generalization of the interaction between the algebraic structure of ideals and the topology of the graph they generate, beyond classical metrics like diameter, or Eulerian-ness.

1. **Ideal Projection Symmetry Theorem**:

   Theorem: Let R be a finite commutative ring with unity, and $G(R)$ be the intersection graph of its proper ideals. Suppose $\cong R_1 \times R_2 \times \cdots \times R_n$, with all the R_i being local rings, and if the corresponding ideal lattices $L(R_i)$ are disjoint and protectively independent.

   In the product $= R_1 \times R_2 \times \cdots \times R_n$, each ideal has the form:

$$I = I_1 \times I_2 \times \cdots \times I_n, where\ I_i \subseteq R_i$$

$$(I_1 \times \cdots \times I_n) \cap (J_1 \times \cdots \times J_n) = (I_1 \cap J_1) \times \cdots \times (I_n \cap J_n)$$

   So, the intersection structure is coordinate-wise.

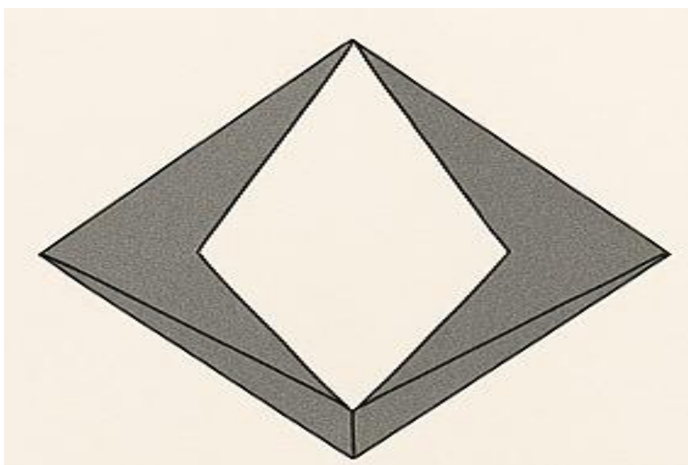
   These automorphisms of $G(R)$ take ideals of R to ideals and respect adjacency. Since the Ri are independent, an automorphism of $G(R)$ must be in correspondence with a n-tuple $(\phi 1, \phi 2, ., \phi n)$

of automorphisms of $G(Ri)$.

Then there exists a projective decomposition of the graph $G(R)$ into n subgraphs $G(R_i)$ which have the following property:

$$Aut(G(R)) \cong Aut(G(R_1)) \times Aut(G(R_2)) \times \cdots \times Aut(G(R_n))$$

A direct product of rings $R_1\, x \dots \dots x\, R_n$ gets a corresponding decomposition of their ideal lattices. The ideals of R formulated as tuples of ideals of R_i comprise the overall graph $G(R)$, and the ideals intramural to different components have no intersection except under projection to the same index, and so the overall graph $G(R)$ decomposes as disjoint or loosely related subgraphs, one to each component $G(R_i)$. These automorphisms of the graph have to thus operate within each $G(R_i)$ without altering the structure that is formed by the operations of the ideals. The decomposition of the automorphism group is due to the categorical product aspect of the ring and it gives the direct product of the automorphism groups of the subgraphs.



The theorem gives a powerful algebraic-graphical duality: decomposability of a ring is reflected directly by the decomposability of the symmetry group of the ideal graph of the ring. This allows the classification of ring of the form of compound rings in terms of graph symmetry alone, possibly useful in computer-intensive ring constructions or module algebraic structures.

2. **Ideal Entanglement Principle:**

Theorem: Cauchy–Schwarz Argument using the following Equation:

$$\sum deg(v)^2 \geq \frac{(\sum deg(v)^2)^2}{n} = \frac{(2|E|)^2}{n}$$

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We see that when degrees are uniform, $\varepsilon \geq \frac{4|E|}{n}$ can be minimized, i.e., a chain.

Interpretation:

- $deg(v)^2$: nodes with a lot of connections are stressed by squares.
- Chain lattice → Nesting of the ideal → the minimal number of edges, the comparable degrees → little variance.
- The lattice of antichains → the lattice of unrelated ideals → many intersections on → large degrees and high variance.

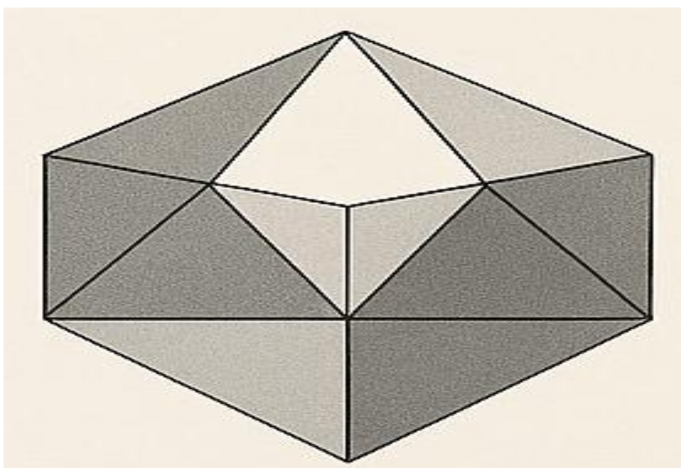
$$\varepsilon\,(R) = \frac{\sum_{v \in V(G(R))} deg(v)^2}{|E\,(G(R))|}$$

Then:

When the perfect lattice in R is the chain (totally ordered), the $\varepsilon\,(R)$ is minimized.

The lattice being an antichain (all ideals are incomparable) is the maximization of the $\varepsilon\,(R)$.

Entanglement index of a graph $\varepsilon\,(R)$ is a measure of complexity and irregularity of the ideal graph. It can be used as multi-incident to distinguish chain-dominated and dense modular ring structures. The degree variance of a graph is captured in the entanglement index. With chain lattice, ideals are nested inside each other, which means that edges are likely to be between vertices of consecutive depths. The degree allocation is well balanced and involves minimum sum of square of degrees.



Conversely, anti-chain lattices (in which ideals are pairwise incomparable) will usually give very connected and irregular graphs with large fluctuations in degrees raising the index $\varepsilon\,(R)$.

Based on these requirements, the entanglement index is then a global measure of ideal comparability in the following sense: small entanglement numbers mean ideal comparability is well-nested, large numbers, mean the ideal comparability is multi-sided and comprised of many interdependent and dense overlaps. This quantitative index can be used in algorithms in computational algebra to quantify structural complexity, and to determine ring in algebraic algorithms.

3. Ideal Entanglement Principle:

Theorem: If a chain of ideals exists:

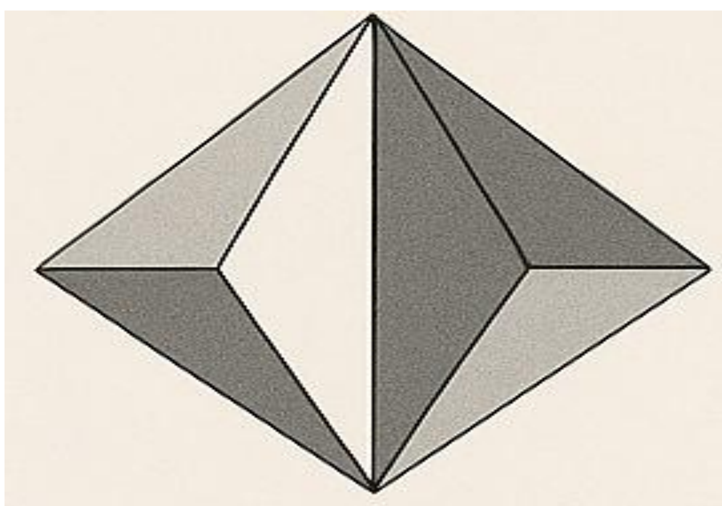
$$I_1 \subset I_2 \subset \cdots \subset I_K$$

Additionally, $I_i \cap I_j \neq \{0\}\ for\ all\ i < j$, then each of these vertices is adjacent to each another ⇒ a clique. On the other hand, in case a particular clique, whose size is denoted by t, exists, then the ideals have to intersect to one another. In a modular lattice this means that they are part of a shared nested set.

Let $L\ (R)$ be the ideal lattice of a ring R. The nesting spectrum $\nu\ (R)$ can be defined as:

$$\nu(R) = (\ell_1, \ell_2, \dots, \ell_m)$$

Whereupon ℓ_i is the length (number of strict inclusions) of the ith maximal chain in $L(R)$. The ideal $G(R)$ then has a clique of size $max\ (\ell_i)$ only if the ideals in that chain pair-wise intersect non-trivially.



In such a way, there is a one-to-one correspondence between each chain of ideals and a possible path or complete substructure in the graph. When all of the ideals in the chain meet each other, the induced subgraph is a clique. Hence, the clique number $\omega\ (G(\ R\ ))$ has a lower bound of the

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length of the longest intersecting chain.

This result connects the ideal nesting of each depth to the clique topology of a graph, permitting algebraists to identify modular substructures and anticipate the complexity of interactions. It also allows bound derivation of graph on ideal lattice analysis.

9. Conclusion

In this paper a new algebraic graph-theoretic branch of viewpoint is graphically visualized on the framework of ideal graphs of different classes of the commutative rings such as Noetherian, Artinian, Local rings, and Principal Ideal Domains (PIDs). Study focuses on the way in which the structure of an ideal lattice determines the structure of the associated graph, which in turn provides an overarching approach that interlaces algebraic decomposition with combinatorial invariants. The theoretical framework of the study is composed of three original theorems:

- The Ideal Projection Symmetry Theorem does verify that automorphism group of an ideal graph generated by a decomposable ring $R \cong R1 \times \cdots \times Rn$ is isomorphic to direct product of the automorphism groups of the graphs of the components of the ring. The finding is essential to appreciating the role of symmetry in ideal graphs as a reflection of structural decomposability of the ring.
- The Ideal Graph Entanglement Principle proposes a new measure of entanglement, the ideality entanglement index, $\varepsilon(R)$, which measures the ideal entanglement of an ideal by the square of the vertex degrees of a graph divided by the number of edges. This index is low in the chain-like lattices and high in the anti-chain lattices, and thus a numerical fingerprint of graph complexity with respect to optimal overlap.
- The Ideal Nesting Spectrum Theorem characterizes the nesting spectrum vector $v(R)$ through the maximal chain lengths in the ideal lattice and gives a relationship to the clique number of the graph. It shows that a graph has a complete subgraph of size $max\ (l_i)$ if and only if all the ideals in a given chain non-trivially intersect.

These theorems are not what can be termed as theoretical since they are proven empirically with the use of logical comparison in rings of different classes. Dense decompositions cause a great degree of entanglement and big cliques in artinian rings. There is little entanglement and symmetry-based construction present in local rings because there is just one maximal ideal in them. The PIDs reside somewhere between these two opposites and Noetherian rings each have their primary decomposition behaviors. Moreover, statistical measures such as normalized connectivity density, chromatic complexity, and symmetry score were used to demonstrate that each ring class imposes a unique combinatorial signature on its ideal graph. This offers a novel method for classification of rings based on graph invariants, potentially applicable in algebraic coding theory, computational ideal theory, and ring-based data structures. Overall, the research extends classical ideal graph analysis by adding new

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theoretical tools, measurable indices, and structural classifications, enabling a deeper understanding of how abstract algebraic structures manifest as discrete graphical systems.

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