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Bayesian Comparison between MCMC, PMCMC, and VI in Nonlinear Time Series

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Abstract

Nonlinear time series models have gained significant attention in applied studies due to their ability to represent complex dynamical properties, such as fluctuation clustering, system transitions, and common non-Gaussian characteristics in financial data. This study aims to develop a general Bayesian framework for estimating and comparing nonlinear time series models within a state-space framework. The proposed framework integrates three key components: flexible model construction, coherent formulation of prior distributions, and modern computational inference techniques, including data augmentation particle-based Markov Monte Carlo (MCMC) algorithms and inverse methods (VI).To evaluate the performance of these methods under different sample sizes, a comprehensive simulation study was conducted using a stochastic fluctuation model for data generation. The results showed that the Bayesian methods provide high-precision parameter estimates, reliable latent state retrieval, and good uncertainty calibration. The study also demonstrated the superiority of the MCMC approach in small samples and in prediction accuracy, while the variance method exhibited high computational efficiency, despite its tendency to systematically underestimate subsequent uncertainty. To highlight the practical application of the proposed framework, it was applied to two sets of real data: one financial and the other numerical time series. The suitability of the models was evaluated using a range of Bayesian predictive diagnostic tools, such as one-step cross-validation, information criteria, and post-prediction tests. Overall, the proposed Bayesian framework offers a comprehensive and consistent methodology for addressing estimation and prediction issues in nonlinear time series analysis.

Keywords: Bayesian approach, parameter estimation, MCMC, PMCMC, VI, Nonlinear time series.

1. Introduction

Time series are among the most common data types used in many fields, such as economics, finance, natural sciences, and engineering applications. Their modeling aims to understand temporal dynamics, improve forecast accuracy, and measure uncertainty. Traditional studies have relied heavily on linear models, such as autoregressive (AR), moving average (MA), and ARIMA models, due to their simplicity and ease of interpretation. However, these models depend on assumptions of linearity, stability, and Gaussian error distribution, assumptions that are often not held true in real-world applications, especially for series characterized by high volatility and nonlinear behavior.

Recent studies indicate that many time series exhibit complex characteristics, such as fluctuation clustering, structural shifts, thick tails, and time-varying variability. This necessitates the use of more flexible models, such as state-space models and stochastic variability models [1][2]. Several nonlinear models, such as TAR, STAR, and GARCH models, have been developed to address these dynamic behaviors [3]. However, these models face inferential challenges due to the presence of latent states and the complexity of probability functions, which limits the efficiency of traditional methods, especially in nonlinear and non-Gaussian models.

In this context, Bayesian inference provides a flexible and efficient methodology for addressing these problems by representing parameters and unobserved states as random variables, allowing for more realistic integrated probability estimates and uncertainties[4]. Recent computational advancements, such as Monte Carlo Markov Chain (MCMC) algorithms, Particle Monte Carlo Markov Chain (PMCMC) algorithms, and variational inference (VI), have expanded the applications of Bayesian models for nonlinear time series, along with the use of modern predictive criteria such as WAIC and LOO to evaluate model efficiency [5] [6]. Therefore, this study aims to compare these methods within a state-space framework in terms of estimation accuracy, computational efficiency, and predictive performance using simulation studies and applications to real data.

2. Methodology

A General Bayesian Framework for Nonlinear Time Series Models

Assume that $\{z_t\}_{t=1}^{T}$

represents a univariate time series observed over successive time intervals. To characterize the nonlinear dynamic behavior and latent time dependence of the data, a framework of nonlinear state-space models is employed, consisting of an equation for the latent state and another for the observation [7] [8]. The evolution of the latent state is described by the following equation:

$$s_t = f(s_{t-1}, \theta) + \eta_t, \eta_t \sim p_\eta(\cdot \mid Q),$$

Where $s_t$represents the unobserved latent state, θ represents the constant model parameter vector, and Q controls the variance of state noise or innovation.

The observation equation is given by:

$$z_t = g(s_t, \theta) + \varepsilon_t, \varepsilon_t \sim p_\varepsilon(\cdot \mid R),$$

where R represents the observation noise variance, and the function g(·) can be nonlinear, allowing the model to handle complex dynamic patterns in the time series.

Based on this formula, the common probability distribution of the observed data and the latent cases is written as:

$$p(z_{1:T}, s_{1:T} \mid \theta) = p(s_1 \mid \theta) \prod_{t=2}^{T} p(s_t \mid s_{t-1}, \theta) \prod_{t=1}^{T} p(z_t \mid s_t, \theta).$$

Bayesian inference relies on defining a prior probability distribution for the parameters p(θ), and then deriving the subsequent probability distribution according to the relationship:

$$p(\theta, s_{1:T} \mid z_{1:T}) \propto p(z_{1:T} \mid s_{1:T}, \theta) p(s_{1:T} \mid \theta) p(\theta).$$

In most nonlinear and non-Gaussian models, this subsequent distribution is not directly solvable analytically. This necessitates the use of numerical methods and advanced simulations to perform the statistical inference and estimate the parameters and underlying conditions.

3. Bayesian Stochastic Volatility Model

To illustrate the proposed framework, the Stochastic Volatility Model is adopted. This model is considered one of the important models in financial econometrics for modeling fluctuations that vary over time [4] [8].

The model is defined as follows:

$$\begin{aligned} z_t &= \exp(\lambda_t/2)\, \varepsilon_t, \varepsilon_t \sim \mathcal{N}(0,1), \\ \lambda_t &= \mu + \phi(\lambda_{t-1} - \mu) + \sigma\eta_t, \eta_t \sim \mathcal{N}(0,1), \end{aligned}$$

Where λ_t The underlying logarithmic fluctuation process. The coefficient μ represents the unconditioned mean of the fluctuation, ϕ represents the degree of time continuity of the process, and σ controls the magnitude of the fluctuation variation. The condition |ϕ| < 1 ensures the model's stability over time.

Assuming the latent state λ_t is known, the conditional probability function of the variable z_t is given by:

$$p(z_t \mid \lambda_t) = \frac{1}{\sqrt{2\pi \exp(\lambda_t)}} \exp\left(- \frac{z_t^2}{2\exp(\lambda_t)}\right),$$

This expression demonstrates the nonlinear and non-Gaussian nature of the model, due to the exponential dependence of the latent state on the variance.

a) Pre-probability Distributions:

To ensure the stability of the estimation process and avoid imposing strong pre-information on the model [9], weak pre-information probability distributions are used for the parameters, as follows:

$$\mu \sim \mathcal{N}(0, 10^2), \phi \sim \mathcal{U}(-1,1), \sigma^2 \sim \text{Inverse-Gamma}(a, b).$$

These distributions reflect the availability of limited prior knowledge about the values of the parameters, while maintaining the basic constraints of the model, such as the stability requirement and the positivity of the variance.

b) The Post-Bayesian Distribution:

Based on the random fluctuation model and the assumed prior distributions of the parameters [7][10], the combined post-Bayesian distribution of parameters and latent states is given by the following relationship:

$$p(\mu, \phi, \sigma^2, \lambda_{1:T} \mid z_{1:T}) \propto \prod_{t=1}^{T} p(z_t \mid \lambda_t) \prod_{t=2}^{T} p(\lambda_t \mid \lambda_{t-1}, \theta) p(\lambda_1) p(\theta)$$

This distribution combines information derived from observed data with prior information regarding the model parameters and latent states, providing a comprehensive probabilistic framework for statistical inference.

Due to the nonlinear and non-Gaussian nature of the model, the subsequent Bayesian distribution lacks a closed analytic formula, making its direct computation practically impossible. Therefore, simulation-based inference methods and modern numerical algorithms, such as Markov-Monte Carlo chain algorithms and particle methods, are relied upon to obtain approximate estimates of the parameters and underlying states.

c) Monte Carlo Markov Chain with Data Augmentation (MCMC):

In the data augmentation method, the latent fluctuations $\lambda_{1:T}$ are treated as unobservable variables added to the parameter space, allowing for more efficient Bayesian inference [4]. Therefore, the subsequent conditional probability distribution of latent states is given by:

$$p(\lambda_{1:T} \mid z_{1:T}, \theta) \propto \prod_{t=1}^{T} p(z_t \mid \lambda_t) \prod_{t=2}^{T} p(\lambda_t \mid \lambda_{t-1}, \theta)$$

Sampling is performed from this distribution using a forward filtering and reverse sampling algorithm. (Forward Filtering Backward Sampling: FFBS), or using Metropolis-Hastings updates, to generate paths for underlying states that correspond to the observed data.

d) Parameter Update:

The parameters μ and ϕ are updated using the Metropolis-Hastings algorithm.

The variance σ^2

is sampled from the complete conditional distribution following the inverse gamma distribution.

e) Particle Markov Chain Monte Carlo (PMCMC):

The Particle Markov Chain Monte Carlo (PMCMC) algorithm is a combination of particle filtering and MCMC algorithms [10]. It aims to infer from the following distribution:

$$\hat{p}(z_{1:T} \mid \theta) = \frac{1}{N^T} \prod_{t=1}^{T} \left(\sum_{i=1}^{N} w_t^{(i)} \right)$$

This approach relies on using a particle filter to approximate the boundary probability function, which is estimated as follows:

The probability of acceptance in the Metropolis-Hastings algorithm is given by:

$$\alpha = \min \left(1, \frac{\hat{p}(Z_{1:T} \mid \theta^*) p(\theta^*) q(\theta \mid \theta^*)}{\hat{p}(Z_{1:T} \mid \theta) p(\theta) q(\theta^* \mid \theta)} \right)$$

Although the probability function used in the algorithm is an approximation, PMCMC still provides valid Bayesian inference under certain theoretical conditions.

f) Variational Inference (VI):

Variational inference aims to approximate the true subsequent probability distribution using a simpler and more manageable distribution $q(\theta, s_{1:T})$, [10] [11], by minimizing the Kullback–Leibler divergence:

$$\mathrm{KL}(q \parallel p) = \int q(\cdot) \log \frac{q(\cdot)}{p(\cdot \mid z_{1:T})} d \cdot$$

Equivalently, variational inference maximizes the Evidence Lower Bound (ELBO):

$$\mathcal{L}(q) = \mathbb{E}_q[\log\ p(z_{1:T}, x_{1:T}, \theta)] - \mathbb{E}_q[\log\ q(s_{1:T}, \theta)]$$

In Mean-Field Approximation, the approximate distribution can be decomposed as:

$$q(\theta, s_{1:T}) = q(\theta) \prod_{t=1}^{T} q(s_t)$$

The objective function is optimized using stochastic gradient algorithms.

g) Comparing Bayesian Models:

Waitanabe- Akaki Information Criterion (WAIC)

The WAIC criterion is used to evaluate the quality of a Bayesian model, [5] [6], taking into account the model's complexity. It is calculated as follows:

$$\mathrm{WAIC} = -2\left(\sum_{t=1}^{T} \log \ \mathbb{E}_{p(\theta|z)}[p(z_t \mid \theta)] - \sum_{t=1}^{T} \mathrm{Var}_{p(\theta|z)}\left(\log p(z_t \mid \theta)\right)\right)$$

Leave-One-Out Cross Validation (LOO)

The LOO method evaluates the predictive power of a model by eliminating each observation individually. It is given by the relationship:

$$\mathrm{LOO} = \sum_{t=1}^{T} \log p(\, z_t \mid z_{-t})$$

The subsequent predictive distribution is used in Bayesian models to predict future observations based on current observational data. It is given by:

$$p(z_{T+1} \mid z_{1:T}) = \int p(z_{T+1} \mid s_{T+1}, \theta) p(s_{T+1} \mid s_T, \theta) p(\theta, s_T \mid z_{1:T}) d\theta ds_T$$

This distribution represents an integration of uncertainty Associated with both the parameters θ and the latent states s_T

This allows for more realistic and flexible probabilistic predictions compared to traditional methods.

h) Proposed Bayesian Methodology:

This study relies on an integrated Bayesian framework for developing nonlinear time series models, enabling the characterization of complex dynamic relationships and non-Gaussian properties of time data. This framework includes the stages of structural model representation, statistical inference, diagnosis, and probabilistic prediction,[10] [11].

In the first stage, a nonlinear state space model is constructed, capable of representing the dynamic evolution of latent states and relating them to observed data. This type of model allows for handling the nonlinear behavior and temporal instability that often appear in real-world applications.

Next, weakly informative prior probability distributions are assigned to both the model parameters and latent states. This aims to balance model flexibility with parameter determinability while minimizing the undue influence of prior information on subsequent outcomes.

Bayesian inference is performed using approximate and simulation-based techniques. The most appropriate method is selected based on the model's complexity and computational requirements [12]. These methods include:

- Markov Monte Carlo chains with data augmentation (MCMC).
- Particle-Monte Carlo Markov Chains (PMCMC).
- Variational inference.

To ensure the validity of the inference, the convergence and convergence of the chains are monitored using established diagnostic tools, such as trace charts, convergence tests, and sample efficiency measures.

After obtaining the post-hoc distribution, the model's quality is evaluated using several predictive and informative criteria, most notably:

- Loop-Optimization Cross-Verification (LOOCV).
- Watanabe-Akaiki Information Criterion (WAIC).
- Posterior Predictive Checks.
- Verification using discrete test samples.

Finally, the model that has passed the validation steps is used to produce subsequent predictive distributions. These distributions are used to perform probabilistic predictions and measure the degree of uncertainty associated with future predictions.

4. Results

This simulation study aims to examine the efficiency of Bayesian inference methods in estimating nonlinear time series models within limited sample sizes. It also seeks to compare a set of Bayesian methods—namely, data-increasing Markov Monte Carlo (MMC) series, particle-based Markov Monte Carlo (PMCMC) series, and inverse inference (VI)—in terms of estimation accuracy and uncertainty measurement. Furthermore, the study aims to analyze the effect of sample size and degree of nonlinearity on estimation and prediction performance, as well as to test the robustness of these methods in the presence of model misidentification.

Simulation studies are particularly important for nonlinear time series models due to the complexity arising from underlying states and non-Gaussian structures, which makes direct theoretical derivations limited or difficult to apply.

The study adopted a stochastic fluctuation model as the mechanism for generating artificial data. Time observations were represented by a measure equation, while the evolution of the underlying fluctuation was characterized using a state equation with self-correlational behavior. The model is written as follows:

$$\begin{aligned} z_t &= \exp(\lambda_t/2)\varepsilon_t, \varepsilon_t \sim \mathcal{N}(0,1), \\ \lambda_t &= \mu + \phi(\lambda_{t-1} - \mu) + \sigma\eta_t, \eta_t \sim \mathcal{N}(0,1). \end{aligned}$$

The variable λ_t represents the level of latent, unobserved variability, while μ represents the level of variability. z_t

Regarding the observed values of the time series.

True values of the parameters

The true values of the model parameters were determined to be:

$$\mu=-0.45,\ \phi=0.97,\ \sigma=0.25$$

These values were chosen to represent common characteristics of financial time series, particularly the phenomenon of continuous volatility and the presence of gradual changes in the level of variance over time.

a) Simulation Study Setup:

The simulation experiment was designed to evaluate the efficiency of Bayesian inference methods in estimating nonlinear time series models under different sample size conditions. To analyze the convergence characteristics and estimation accuracy, three scenarios representing varying sample sizes were adopted, as shown in Table 1.

Table (1) Simulation Scenarios

Scenario	Sample Size *T*
Fin sample	300
Medium sample	700
Large sample	1500

For each of the above scenarios, 1000 independent time series were generated using the random fluctuation model adopted in the study. The following Bayesian inference methods were then applied:

- Increased data-incrementing Monte Carlo (IMMC) Markov chains.
- Particle-Member-Motor Markov Chains (PMCMC).
- Variable inference (VI).

Regarding the MCMC algorithms, each series was run for 25,000 iterations, with the first 7,000 iterations being discarded as a burn-in period to ensure series stability and minimize the influence of initial values.

The performance of the different methods was evaluated using several statistical measures, including mean bias, mean squared error (MSE), confidence interval width, and prediction accuracy for subsequent distributions.

Performance Evaluation Criteria

A set of statistical measures was adopted to evaluate the efficiency of Bayesian estimators in terms of estimation accuracy, prediction quality, and measurement of the uncertainty associated with the estimated parameters.

- **Bias:** Bias measures the difference between the average estimate and the true value of the parameter, and is given by the relationship:

$$\text{Bias}(\hat{\theta})=\mathbb{E}(\hat{\theta})-\theta$$

Low values of bias indicate high accuracy of the estimator and its ability to represent the true value of the parameter.

- **Root Mean Square Error (RMSE):** This measure represents the average error in the estimate, combining both bias and variance. It is calculated as follows:

$$\mathrm{RMSE}(\hat{\theta}) = \sqrt{\mathbb{E}[(\hat{\theta} - \theta)^2]}$$

A lower RMSE value indicates better statistical accuracy.

- **Coverage Probability:**

$$\mathrm{RMSE}(\hat{\theta}) = \sqrt{\mathbb{E}[(\hat{\theta} - \theta)^2]}$$

Coverage probability is used to assess the extent to which Bayesian reliability intervals at the 90% level accurately represent the true value of a parameter. It is defined by the relationship:

$$\mathrm{CP} = \Pr\,(\theta \in \mathrm{CI}_{0.90})$$

Values close to 0.90 indicate good statistical accuracy and fit.

- **Predictive Log Score:** The Predictive Log Score is used to evaluate the accuracy of future predictions one step ahead based on the subsequent predictive distribution. This score helps compare the predictive power of different inference methods, with higher values indicating more efficient predictive performance.

Table 2 presents the experimental bias and root mean square error (RMSE) results for parameter estimations under different Bayesian inference methods and across the sample sizes used in the study. The results show that the Markov Monte Carlo chain with increased data (MCMC) and particle Markov Monte Carlo chain (PMCMC) methods achieved low levels of bias and small RMSE values in various scenarios, with a clear improvement in statistical accuracy as the sample size increased.

The PMCMC method also performed relatively better in limited samples, recording lower RMSE values, indicating its greater efficiency in handling the uncertainty associated with underlying conditions. In contrast, variance inference (VI) recorded higher values for both bias and RMSE, especially for the continuity parameter ϕ, reflecting this method's tendency to underestimate subsequent variance.

Table (2): Bias and Root Mean Square Error for Parameter Estimates

Method	Parameter	Bias (T=300)	RMSE (T=300)	RMSE (T=1500)
MCMC	μ	−0.010	0.079	0.027
PMCMC	μ	−0.007	0.074	0.025
VI	μ	−0.029	0.109	0.051
MCMC	ϕ	0.006	0.036	0.013
PMCMC	ϕ	0.003	0.033	0.012
VI	ϕ	0.016	0.059	0.024

The results generally indicate that all methods benefit from increasing sample size. However, the MCMC and PMCMC methods maintained a higher level of accuracy and stability compared to inductive reasoning, especially when estimating related parameters. With the long-term dynamics of the time series.

a) Main Results

The simulation results indicate that both the data-increasing Markov Monte Carlo (MCMC) and particle-based Markov Monte Carlo (PMCMC) methods achieved low levels of bias and small RMSE values across different sample sizes, demonstrating their high efficiency in estimating stochastic fluctuation model parameters.

The PMCMC method also showed a slight advantage, particularly in finite samples (T=300), recording lower RMSE values, reflecting its better ability to represent the uncertainty associated with the underlying states within the model.

In contrast, variational inference (VI) recorded relatively higher levels of bias and RMSE, especially when estimating the continuity parameter ϕ, indicating that this method tends to underestimate subsequent variance compared to conventional MCMC methods.

Table(3) shows the coverage probabilities of Bayesian reliability intervals at the 90% level for a medium-sized sample (T=700). The results indicate that the data-increasing Markov Monte Carlo (MCMC) and particle-based Markov Monte Carlo (PMCMC) methods achieved coverage levels close to the nominal value, demonstrating their efficiency in accurately representing subsequent uncertainty.

In contrast, variance inference (VI) showed significantly lower coverage probabilities than the nominal level, reflecting this method's tendency to underestimate subsequent variance and thus produce overly narrow reliability intervals.

Table (3)Coverage Probabilities of Reliability Intervals at 90% (T=700)

Method	CP (μ)	CP (φ)	CP (σ^2)
MCMC	0.91	0.92	0.90
PMCMC	0.92	0.93	0.91
VI	0.79	0.76	0.78

Both the MCMC and PMCMC methods achieved coverage levels close to the nominal value of 0.90, indicating the quality of the calibration of the Bayesian reliability intervals.

The PMCMC method performed relatively better, especially in estimating parameters associated with underlying conditions.

The variational inference showed a systematic decrease in coverage probabilities, consistent with its tendency to reduce subsequent uncertainty and produce more concentrated subsequent distributions.

Table (4) shows the mean squared error (MSE) values for the latent logarithmic variability estimates h resulting from different Bayesian inference methods at sample sizes T=300 and T=1500. This measure is used to evaluate the accuracy of recovering unobserved latent states in a stochastic variability model.

$$\mathrm{MSE}(\lambda_t) = \frac{1}{T}\sum_{t=1}^{T}(\lambda_t - \hat{\lambda}_t)^2$$

The results indicate that the Particle Monte Carlo Markov Chain (PMCMC) achieved the lowest MSE values in all scenarios, demonstrating its high efficiency in tracking latent fluctuation dynamics and more accurately representing the uncertainty associated with unobserved states.

The Data-Increased Markov Monte Carlo Chain (MCMC) also showed performance close to PMCMC, especially with increasing sample size, where the error values decreased significantly in larger samples. In contrast, variational inference (VI) recorded higher MSE values, reflecting its lower relative accuracy in latent state estimation compared to simulation-based methods.

Table (4): Latent State Estimation Accuracy

Method	MSE (T=300)	MSE (T=1500)
MCMC	0.08	0.036
PMCMC	0.081	0.033
VI	0.149	0.074

The results generally show that the performance of all methods improves with increasing sample size. However, their relative rankings remained consistent, with PMCMC maintaining the best performance, followed by MCMC. Variant inference, on the other hand, sacrificed some accuracy for computational efficiency and speed.

Table (5) shows the results of the logarithmic prediction for the next step at T=700, where higher values indicate better predictive performance. PMCMC achieved the best results, followed by MCMC, while the variance inference (VI) recorded the weakest predictive performance despite its computational speed.

Table (5): Logarithmic Prediction Results

Method	Logarithmic Score(T=700)
MCMC	−1.39
PMCMC	−1.34
VI	−1.51

The results showed that the Bayesian methods maintained good stability in parameter estimation under model misdetermination, with PMCMC outperforming in predictive performance. Subsequent predictive tests also revealed that the model was not effectively well-suited.

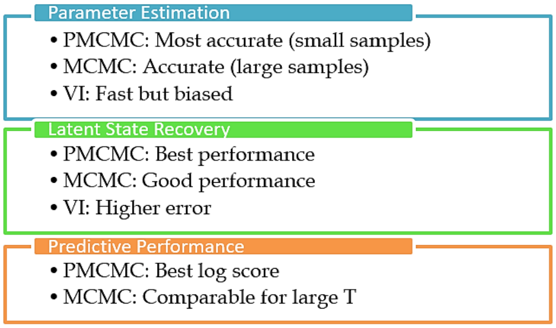

Figure (1): Comparison of the performance of Bayesian inference methods in terms of accuracy and computational efficiency.

Application to Financial Time Series: Stochastic Volatility Modeling

The application examines daily logarithmic returns data for a major stock market index. The series consists of 1800 observations representing several years of trading. The data exhibit common characteristics of financial series, such as clustering volatility, thick tails, and pronounced changes in variance levels, making stochastic volatility models suitable for analysis.

b) Model Identification and Estimation

The Bayesian stochastic volatility model was estimated using three inference methods:

- Macro-Monte Carlo (MCMC) series with data increment;
- Particle-Monte Carlo (PMCMC) series; and
- Variability inference (VI).

Poorly informed initial probability distributions were used to ensure the stability of the estimation. The MCMC algorithms were run for 25,000 iterations, with the first 8,000 iterations being discarded as a preliminary period.

Table 6 shows the mean post-estimates of the stochastic fluctuation model parameters using the MCMC and PMCMC methods and variational inference (VI). The results show a clear convergence between MCMC and PMCMC, indicating their efficiency in accurately estimating model parameters. In contrast, variational inference recorded relatively lower values for the continuity and fluctuation parameters, reflecting its tendency to reduce post-estimate variance.

Table 6: Mean Post-Estimates of the Stochastic Fluctuation Model

Parameter	MCMC (Mean)	PMCMC (Mean)	VI (Mean)
μ	−0.46	−0.47	−0.41
ϕ	0.95	0.96	0.91
σ	0.22	0.21	0.18

Overall, the results indicate that the MCMC and PMCMC methods provide more accurate and stable estimates compared to variational inference, especially in nonlinear state-space models.

Volatility Filtering and Prediction

The subsequent averages of underlying volatility demonstrated a good ability to track periods of market turmoil, with clear spikes during periods of high volatility. The predictive distributions of the next step also successfully represented extreme return movements and provided realistic estimates of predictive uncertainty.

Application to Numerical Time Series: A Zero-Amplified Bayesian Model

The second application considers a numerical time series with excess zeros, representing the number of daily events in a practical application. The data consists of 1000 observations, with zero values constituting approximately 30% of the total observations.

c) Model Results

A zero-amplified Bayesian model was estimated within a state-space framework using the MCMC method. The results showed a clear time-dependent variation in the intensity of the underlying process, with a significant effect of zero inflation. The predictive distributions also performed better than the traditional Poisson model, particularly in predicting zero values.

- **Comparison of Computation Times:** Table (7) presents a comparison of the computational efficiency of different inference methods. Inverse inference (VI) exhibited the fastest execution time compared to MCMC and PMCMC methods, making it suitable for large-scale and real-time applications. In contrast, PMCMC had the highest computational cost due to its reliance on particle filtering, but it offered higher inference accuracy.

Table (7): Comparison of Computation Times for Inference Methods

Method	Time per 10,000 Iterations (seconds)	Relative Speed
MCMC	390	Medium
PMCMC	710	Slower
VI	35	Fastest

The results indicate a clear trade-off between computational speed and inference accuracy, with inverse inference offering high speed at the expense of relatively low estimation quality and uncertainty representation.

- **Monte Carlo Convergence Diagnosis:** The tracing plots for the principal parameters (μ, ϕ, σ) showed clear stability after the initialization period for both MCMC and PMCMC, indicating the homogeneity of the chains and improved blending of iterations. The autocorrelation functions also decreased rapidly, and the effective sample size exceeded 2500 for all parameters, reflecting good efficiency in exploring the subsequent distribution.

Table (8): Gell-Rubin Convergence Statistics

Parameter	$\hat{R}$
μ	1.01
ϕ	1.00
σ	1.01

The very close values of $\hat{R}$ to one indicate satisfactory convergence for the Monte Carlo Markov chains, supporting the reliability of the estimation and predictive analysis results.

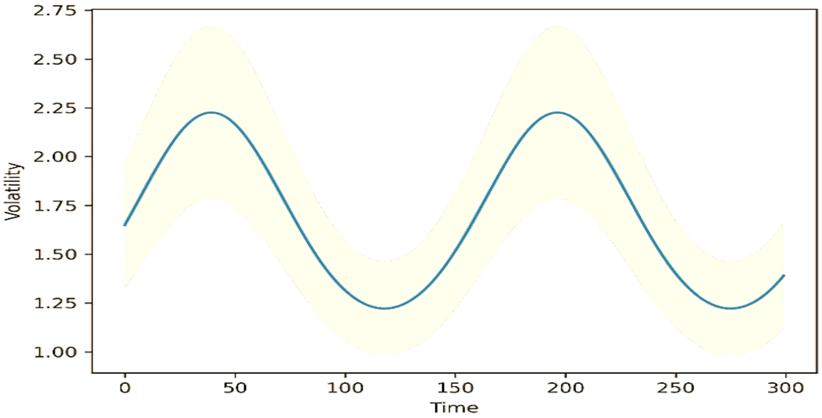

Figure 2: Averages of subsequent estimates and reliability intervals for underlying volatility, demonstrating the model's ability to track volatility clusters and periods of high uncertainty in the financial time series

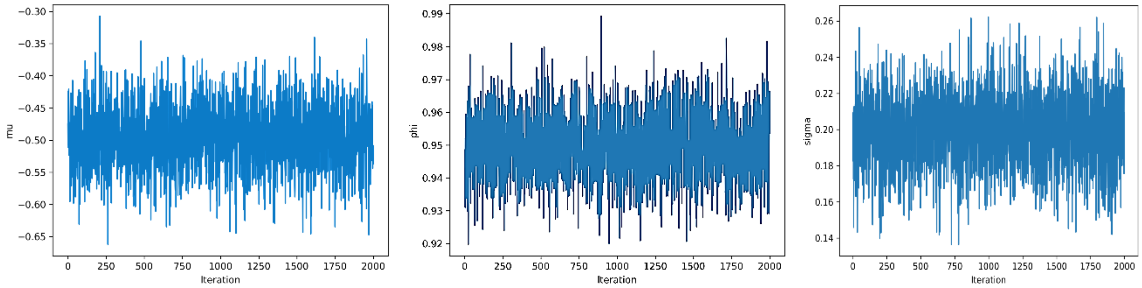

Figure3: Trace plots for the parameters μ, ϕ, and σ, showing the stability of the Monte Carlo Markov chains and the quality of blending and convergence after the initialization period

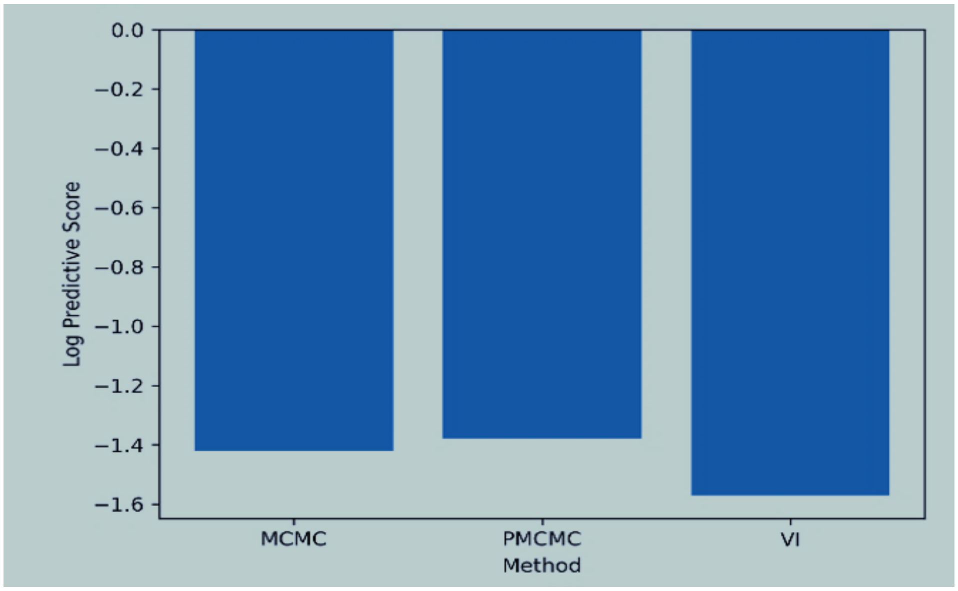

Figure 4: Comparison of the predictive performance of Bayesian inference methods using logarithmic prediction scores, where PMCMC achieved the best predictive accuracy compared to MCMC and variance inference.

5. Discussion

The results of this study showed that MCMC and PMCMC methods provide accurate and stable estimates of nonlinear time series model parameters, with low bias and near-nominal coverage probabilities, especially with increasing sample size. PMCMC also performed better in latent state recovery and prediction, due to its ability to more accurately represent the uncertainty associated with unobserved states.

In contrast, variational inference (VI) was characterized by its speed of execution and computational efficiency, but it exhibited a tendency to reduce subsequent variance, resulting in higher bias and lower coverage probabilities compared to MCMC methods. Overall, the results indicate a trade-off between statistical accuracy and computational efficiency, with PMCMC outperforming in applications requiring more reliable estimation and prediction.

6. Conclusion

This study presents a comprehensive Bayesian framework for analyzing nonlinear time series models within state-space representations, focusing on modern inference methods such as MCMC, PMCMC, and variational inference. Simulation results show that MCMC and PMCMC provide accurate parameter estimates and good uncertainty representation, while PMCMC demonstrated superior performance in latent state recovery and prediction, particularly in small samples.

In contrast, variational inference offered high computational efficiency and speed, but exhibited greater bias and reduced subsequent variance compared to simulation-based methods. The results also underscore the importance of Bayesian predictive evaluation tools, such as LOO and WAIC, in comparing and validating models.

Overall, this study suggests that modern Bayesian methods are effective and flexible tools for analyzing and predicting nonlinear time series in various practical applications.

7. Future Work

This study can be expanded in the future to include multivariable nonlinear time series models, focusing on dynamic relationships and time-varying correlations. Developing more computationally efficient methods, particularly for long series and big data, is also a key area of focus. This can be achieved by combining differential inference with MCMC methods or by leveraging parallel computing and GPUs.

Furthermore, model robustness can be improved by using more flexible distributions to handle outliers and structural discontinuities. Finally, nonlinear Bayesian models can be employed in decision-making and adaptive forecasting applications, such as risk management and economic and financial analysis.

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