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# How to Maximize Minimum Number of Colour Classes for Edge Colouring Using Families of Disjoint Sets Colouring Technique

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## Abstract

In this paper we use families of disjoint sets colouring technique, to introduce some results of edge colouring for minimum and maximum number of colour classes, and introduce a results explains how to find different values of minimum colour classes, and how to maximize minimum number of colour classes for edge colouring, when maximum degree of vertex is fixed, here we prove a result as a method for finding minimum number of colour classes for edge colouring. Families of disjoint sets colouring technique, as generalization technique deal even with graph of multiple edges, and here graph of multiple edges given as application of some examples and results.

**Keywords:** Families of Disjoint Sets Colouring Technique, Minimum Number of Colour Classes, Maximum Number of Colour Classes.

## I. Introduction

Families of disjoint sets colouring technique is a trial to unify all types of partitioning and colouring, and is a trial to generalize all types of partitioning and colouring, (edges, vertices, faces and any things can be contrast by colours). Families of disjoint sets colouring technique depends generally on the number $n$, in case of sets $n$ is degree of intersection, for colouring edges of the graph $n$ is the largest degree of vertex. For colouring vertices of the graph, $n$ is the maximum clique number (maximum number of vertices such that any two vertices are adjacent) [10].

In paper [3] we introduced the concept of families of disjoint sets colouring technique, and published papers [4],[5],[6],[7],[8],[9],[10,to introduce some results related to this colouring (partition) technique.

In paper [11], we introduce the concept of common set to establish some results for families of disjoint sets colouring technique for partitioning sets. The concept of common set is essential concept for families of disjoint sets colouring technique, and this was reason for modifications of some definitions and some results in previous papers.

Paper [12] analogue to paper [11]. In paper [12], we introduce the concept of common edge between two vertices, as essential concept for edge colouring to establish some results of edge colouring.

In paper [13], we introduce concept of common vertex and quasi non common vertex, and we introduce some results related to these concepts. Also in Paper [13], we introduced a result for method used to partition vertices into minimum number of colour classes.

In section two of this paper, we recall some definitions and results related to families of disjoint sets colouring technique which will be needed.

In section three of this paper, we modify definitions of minimum number of colour classes, definitions of maximum number of colour classes and definitions of standard of partition, and then introduce a result for number of colour classes. In section four we prove a result for method of finding minimum number of colour classes for edge colouring. In section five we explain how to find different values of minimum colour classes, for edge colouring, when maximum degree of vertex is constant, and introduce a condition to maximize minimum number of colour classes. Families of disjoint sets colouring technique, as generalization technique deal even with graph of multiple edges, in this section we introduce some examples of multiple edges given as application of some results include simple graph and multiple graphs.

## II. Related Work

Definitions and results in this section will be needed through this paper.

Definition 2.1. Let $G$ be a graph consisting of nonempty set $V(G)$ of vertices and nonempty set $E(G)$ of edges, an edge $e$ is called common edge between two common vertices $v_i$ and $v_j$ if $e$ joins $v_i$ and joins $v_j$ such that $v_i$ link to $v_1$ and link to $v_2$, and $v_j$ link to $v_3$ and link to $v_4$, where $v_1 \neq v_2$ and link to $v_3 \neq v_4$. [12].

Remark 2.2. As abbreviation instate of common edge between two common vertices, we say common edge [12].

Definition 2.3. Let $G$ be a graph consisting of nonempty set $V(G)$ of vertices and nonempty set $E(G)$ of edges, an edge $e$ is called non common edge between two common vertices $v_i$ and $v_j$ if $e$ joins $v_i$ and joins $v_j$ such that at least one of $v_i$ and $v_j$ is of degree one [12].

Definition 2.4. Let $G$ be a graph of $k$ common edges $e_1, e_2, e_3, ..., e_k$ and these $k$ common edges intersect at $w$ proper vertices $v_1, v_2, v_3, ..., v_w$ (standard of partition), with maximum degree of vertex equal $n$, if these $k$ edges partition into $m$ colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, then $m$ (where $n+1 \leq m$,) is called minimum number of colour classes if we try to partition these $k$ edges into $m-1$ colour classes, there exists a colour class $\chi_i$ such that $w+1 \leq t_i$, where $1 \leq i \leq m-1$, and $t_i$ appearance of $\chi_i$ at $v_1, v_2, v_3, ..., v_w$ [12].



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Definition 2.5. Let $G$ be a graph of $k$ common edges $e_1, e_2, e_3, ..., e_k$ and these $k$ common edges intersect at $w$ proper vertices $v_1, v_2, v_3, ..., v_w$ (standard of partition), with maximum degree of vertex equal $n$, if these $k$ edges partition into $m$ colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, then $m$ is called maximum number of colour classes, if we partition these $k$ edges into any $m+1$ colour classes, there exist two colour classes $\chi_i, \chi_j$ such that $t_i + t_j \le w$, where $1 \le i < j \le m+1$, and $t_i, t_j$ appearance of $\chi_i, \chi_j$ at $X_1, X_2, X_3, ..., X_w$, respectively [12].

Definition 2.6. Let $G$ be a graph of $k$ common edges $e_1, e_2, e_3, ..., e_k$ and these $k$ common edges intersect at $w$ proper vertices $v_1, v_2, v_3, ..., v_w$ with maximum degree of vertex equal $n$, if these $k$ edges partition into $m$ colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, then these $w$ vertices is called standard of partition these $k$ sets into $m$ colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, if for any two colour classes $\chi_i$ and $\chi_j$ where $1 \le i < j \le m$, then $w+1 \le t_i + t_j$ and then $t_i \le w$, $t_j \le w$ [12].

Definition 2.7. Let $G$ be a graph of $k$ common and non-common vertices $v_1, v_2, v_3, ..., v_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$ ), let these $k$ vertices partition into $m$ colour classes, then these $m$ colour classes are called minimum colour classes, if whatever we try to these $k$ vertices into $m-1$ colour classes, there exist at least two adjacent vertices belong to the same colour class [13].

Definition 2.8. Let $G$ be a graph of $k$ common and non-common vertices $v_1, v_2, v_3, ..., v_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$ ), let these $k$ vertices partition into $m$ colour classes, then these $m$ colour classes are called maximum colour classes, if whatever we try to partition these $k$ vertices into $m+1$ colour classes, there exist two colour classes, such that any two vertices belong to these two colour classes, are disjoint [13].

Definition 2.9. Let $G$ be a graph of common and non-common vertices $v_1, v_2, v_3, ..., v_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$), these $k$ vertices partition into $m$ colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, if $w^*$ be number of proper cliques $C_1^*, C_2^*, C_3^*, C_{w^*}^*$ of common vertices $v_1^*, v_2^*, v_3^*, ..., v_{k^*}^*$ satisfies $w^* + 1 \le t_i + t_j$, for any two colour classes $\chi_i$ and $\chi_j$, then the cliques $C_1^*, C_2^*, C_3^*, C_{w^*}^*$ called standard of partition these $k^*$ common vertices into minimum $m$ colour classes, where $t_i$ and $t_j$ be number of times the colour classes $\chi_i$ and $\chi_j$ appears at these $w^*$ cliques respectively, $V^* \subseteq V$, $\Psi^* \subseteq \Psi$, $k^* \le k$, $w^* \le w$, $V = \{v_1, v_2, v_3, ..., v_k\}$, $V^* = \{v_1^*, v_2^*, v_3^*, ..., v_{k^*}^*\}$, $\Psi = \{C_1, C_2, C_3, ..., C_w\}$, $\Psi^* = \{C_1^*, C_2^*, C_3^*, ..., C_{w^*}^*\}$, $1 \le i < j \le m$ [13].

Proposition 2.10. Suppose we have common edges $e_1, e_2, e_3, ..., e_k$ and $v_1, v_2, v_3, ..., v_w$ are proper vertices of these $k$ edges, with maximum degree of vertex equal $n$, suppose these $k$ edges and non-common edges partitioned into $m$ colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, labelled by $1,2,3,..., m$, and if $t_i$ be number of times the colour class $\chi_i$ appear at these $w$ vertices, and for all $1 \le i \le m$, $1 \le l \le k$, $1 \le j \le w$ and $k_1$ is number of non-common edges, then we have $\sum_{i=1}^{m} t_i = k_1 + \sum_{i=1}^{k} w_i = k_1 + \sum_{i=1}^{k} c_i = k_1 + 2k = \sum_{j=1}^{w} d(v_j)$. [12].

Proposition 2.11. Let $G$ be a graph of $k$ common edges $e_1, e_2, e_3, ..., e_k$ and these $k$ common edges intersect at $w$ proper vertices $v_1, v_2, v_3, ..., v_w$ (each vertex is intersection of at least two common edges), with maximum degree of vertex equal $n$, if all edges (common edges and non-common edges) partition into $m$ colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, labelled by $1,2,3,..., m$, and if $t_i$ be number of times the colour class $\chi_i$ appear at these $w$ vertices, where $1 \le i \le m$, and for all $1 \le j \le w$ then we have

$\sum_{l=1}^{k} c_l + \sum_{l=1}^{k_1} c_l = \sum_{j=1}^{w} d(v_j)$, where $c_l$ appearance of edge $e_l$ at $w$ vertices, $1 \le l \le k, 1 \le l \le k_1, k_1$ number of non-common edges [12].

Proposition 2.12. Suppose we have common edges $e_1, e_2, e_3, ..., e_k$ and $v_1, v_2, v_3, ..., v_w$ are proper vertices of these $k$ edges, with maximum degree of vertex equal $n$, suppose all edges (common edges and non-common edges) partition into $m$ colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, for any two colour classes $\chi_i$ meet $\chi_j$ at these $w$ vertices, if $t_i, t_j$ be number of times the colour classes $\chi_i, \chi_j$ appear at these $w$ vertices respectively, where these $w$ vertices is standard of partition of all edges (common edges and non-common edges) into $m$ colour classes, then we have

(i)$w+1 \le t_i + t_j$ For $w$ odd or even. (ii)$\frac{w+1}{2} \le t_i \le w$ (If there is special case existence of colour class $\chi_i$ such that $2 \le t_i \le \frac{w-1}{2}$ it will be only one colour class) for $w$ odd, where $1 \le i < j \le m$.

(iii)$\frac{w+2}{2} \le t_i \le w$ (If there is special case existence of colour class $\chi_i$ such that $2 \le t_i \le \frac{w}{2}$ it will be only one colour class) for $w$ even, where $1 \le i < j \le m$. (iv) If there exist two colour classes $\chi_i, \chi_j$ such that $t_i \le t_j \le \frac{w-1}{2}$ for $w$ odd or $t_i \le t_j \le \frac{w}{2}$ for $w$ even, then $\chi_i$ and $\chi_j$ they will form only one colour class [12].

Theorem 2.13. The sum of degrees of vertices equals twice the number of edges [1].

Definition 2.14. A clique of a graph is a maximal complete sub graph. A clique number $n$ of a graph $G$ is largest number such that is a subgraph $K_n$ of $G$ [15].

Definition2.15. The degree of vertex $v$ of a graph $G$ is the number of edges incident to $v,$ and is written $d(v)$ or $\rho(v).$ the complete graph $K_p$ has $p$ vertices, is regular and each vertex of degree $p-1$ [14].

Definition2.16. A graph $G$ is called simple if between two distinct edges there is no multiple edges and there is any loop [14].

Definition2.17. A vertex colouring of a graph $G=(V,E)$ is a map $c:V\rightarrow S$ such that $c(v)\neq c(w)$ whenever $v$ and $w$ are adjacent. The element of the set $S$ are called the available colours. For small integer $k$ such that $G$ has a map $k-colouring,$ a vertex colouring $c:V\rightarrow\{1,2,3,...,k\}.$ This $k$ is chromatic number of $G,$ it is denoted by $\chi(G).$ A graph $G$ with $\chi(G)=k$ is called $k-chromatic,$ if $\chi(G)\leq k,$ we called $G$ $k-colourable$ [2].

Definition 2.18. Let $G$ be a graph of $k$ common edges $e_1,e_2,e_3,...,e_k$, these $k$ common edges intersect at $w$ proper vertices $v_1,v_2,v_3,...,v_w$, with maximum degree of vertex equal $n,$ let vertices $v_1,v_2,v_3,...,v_w$ be standard of partition edges (common and non-common edges) into $m$ colour classes $\chi_1,\chi_2,\chi_3,...,\chi_m,$ using the following method: In firs step we try partition edges into maximum number of colour classes, such that each colour class has appearance equal $w,$ (we partition till there is no colour class its appearance equal $w$), suppose maximum number of colour classes each has appearance $w$ is $q_0.$ In second step we try partition remaining edges into maximum number of families of disjoint sets, such that each colour class has appearance equal $w-1,$ suppose maximum number of colour classes each has appearance $w-1$ is $q_1.$ In third step we try partition edges into maximum number of colour classes, such that each colour class has appearance equal $w-2,$ suppose maximum number of colour classes each has appearance $w-2$ is $q_2.$ We continue in this process, till in last step suppose maximum number of colour classes each has appearance $w-s$ is $q_s,$ and we have define $\sum_{i=0}^{s}(w-i)q_i = wq_0 + (w-1)q_1 + (w-2)q_2 + ... + (w-s)q_s = \sum_{j=1}^{w} d(v_j)$ where $0 \le q_i,\ 2 \le w-i \le w,$ this method is called construction of minimum number of colour classes [12].

Proposition 2.19. Let $G$ be a graph of $k$ vertices $v_1, v_2, v_3, ..., v_k$ (common and non-common) and $C_1, C_2, C_3, ..., C_w$ are proper cliques of these $k$ vertices, with maximum cliques number equal $n,$ we use the following method to partition these $k$ vertices into $m$ colour classes. In first step we try to find $q_0$ maximum number of colour classes each of common vertices only, and each class has appearance equal $w$ at $C_1, C_2, C_3, ..., C_w,$ such that it is impossible to be equal $q_0 + 1.$ By impossible we mean there exist at least two common vertices belong to one colour class are adjacent. In second step we try to find $q_1$ maximum number of colour classes each of common vertices only, and each class has appearance equal $w-1$ at $C_1, C_2, C_3, ..., C_w,$ such that it is impossible to be equal $q_1 + 1.$ In third step we try to find $q_2$ maximum number of colour classes each of common vertices only, and each class has appearance equal $w-2$ at $C_1, C_2, C_3, ..., C_w,$ such that it is impossible, to be equal $q_2 + 1.$ In last step we try to find equal $q_s$ maximum number of colour classes each of common vertices only, and each class has appearance equal $w-s$ at $C_1, C_2, C_3, ..., C_w,$ such that it is impossible to be equal $q_s + 1.$ After last step all common vertices partition to $m$ colour classes. If $m = q_0 + q_1 + q_2 + ... + q_s$ then $m$ is minimum number of colour classes [13].

Proposition 2.20. Let $G$ be a graph of $k$ common edges $e_1, e_2, e_3, ..., e_k$, these $k$ common edges intersect at vertices $v_1, v_2, v_3, ..., v_w$, with maximum degree of vertex equal $n,$ if we add only one non common sets, without changing maximum degree of vertices, and without addition of any proper vertex, only degree of one proper vertex changes, then these $k$ common edges and non-common edges can be partitioned into $m$ minimum colour classes [12].

Proposition 2.21. Let $G$ be a graph of $k$ common edges $e_1, e_2, e_3, ..., e_k$, these $k$ common edges intersect at vertices $v_1, v_2, v_3, ..., v_w$, with maximum degree of vertex equal $n$, these edges (common and non-common edges) can be partitioned into $m$ colour classes, if we remove only one non common edge, without changing maximum degree of vertex, and without removal of any proper vertex of $v_1, v_2, v_3, ..., v_w$, then after this removal, common edges and non-common edges partitioned into $m$ minimum colour classes [12].

**III. Minimum Number and Maximum Number of Colour Classes**

In this section we modify Definition 2.1., Definition 2.4., Definition 2.5 and Definition 2.6.

By Definition 3.1., Definition 3.2., Definition 3.3. And Defnition3.4. respectively. There is similarity between these modifications (edge colouring) and Definition 2.7., Definition2.8. And Definition 2.9., (vertex colouring) this similarity explains families of disjoint sets colouring technique, unify all types of colouring and some types of partitioning.

Definition 3.1. Let $G$ be a graph consisting of nonempty set $V(G)$ of vertices and nonempty set $E(G)$ of edges, an edge $e$ is called common edge between two common vertices $v_i$ and $v_j$ if $e$ joins $v_i$ and $v_j$, $i \neq j$, such that $d(v_i) \geq 2$ and $d(v_i) \geq 2$. An edge $e$ is called non common edge between two common vertices $v_i$ and $v_j$ if one of $d(v_i)$ and $d(v_i)$ equal one.

Definition 3.2. Let $G$ be a graph of $k$ common and non-common edges $e_1, e_2, e_3, ..., e_k$ (common between proper vertices $v_1, v_2, v_3, ..., v_w$), let these $k$ edges partition into $m$



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colour classes, then these $m$ colour classes are called minimum colour classes, if whatever we try to partition these $k$ edges into $m-1$ colour classes, there exist at least two adjacent edges belong to the same colour class.

Definition3.3. Let $G$ be a graph of common and non-common edges $e_1, e_2, e_3, ..., e_k$ (common between proper vertices $v_1, v_2, v_3, ..., v_w$), these $k$ edges partition into $m$ colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, if $w^*$ be minimum number of proper vertices $v_1{}^*, v_2{}^* v_3{}^*, ..., v_{w^*}{}^*$ of common edges $e_1{}^*, e_2{}^*, e_3{}^*, ..., e_{k^*}{}^*$, satisfies $w^* + 1 \le t_i + t_j$, for any two colour classes $\chi_i$ and $\chi_j$, then the vertices $v_1{}^*, v_2{}^* v_3{}^*, ..., v_{w^*}{}^*$ called standard of partition these $k^*$ edges into minimum $m$ colour classes, where $t_i$ and $t_j$ be number of times the colour classes $\chi_i$ and $\chi_j$ appears at these $w^*$ respectively, where $k^* \le k$, $w^* \le w$, $V^* \subseteq V$, $E^* \subseteq E$, $E = \{e_1, e_2, e_3, ..., e_k\}$, $E^* = \{e_1{}^*, e_2{}^*, e_3{}^*, ..., e_{k^*}{}^*\}$, $V^* = \{v_1{}^*, v_2{}^*, v_3{}^*, ..., v_{w^*}{}^*\}$, $V = \{v_1, v_2, v_3, ..., v_w\}$, $1 \le i < j \le m$. These $m$ colour classes are called maximum colour classes, if whatever we try to partition these $k$ edges into $m+1$ colour classes, there exist two colour classes $\chi_i$ and $\chi_j$ such that $t_i + t_j \le w^*$.

Proposition 3.4. Let $G$ be a graph of $k$ edges $e_1, e_2, e_3, ..., e_k$ and these $k$ edges joint at vertices each common vertex $v_1, v_2, v_3, ..., v_w$, where w odd, with maximum degree of vertex equal $n$, if these $k$ edges partition into $m$ colour classes, then $n \le m \le 2n - 1$, if $m = 2n - 1$ then we have $2n \le w + 1$.

Proof: Using Theorem 2.13. And Proposition2.12. we can write $\sum_{i=1}^{m} t_i \le wn$ to obtain maximum of colour classes we have minimum value of $t_i$. From definition of maximum degree of vertex we have $n \le m$. If w odd then minimum value of $t_i$ is $t_i = \frac{w+1}{2}$, for all $1 \le i \le m$, if $m \ge 2n$, then $(2n + l)\left(\frac{w+1}{2}\right) \le wn$, where $l =$

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المجلة الدولية للبحوث العلمية
Vol. (3), No. (12) December 2024 الإصدار (3)، العدد (12)

0,1,2,3 ...then we have $2n + wl + l \leq 0,$ a contradiction, therefore $n \leq m \leq 2n - 1$. If $m = 2n - 1$ then$(2n - 1)\left(\frac{w+1}{2}\right) \leq wn$, then $2n \leq w + 1$.

## IV. Method of Finding Minimum Number of Colour Classes

In paper [12] we define this method as method of finding minimum number of colour classes for edge colouring, see definition 2.18., here we introduce the method as a result (Proposition 4.1.). This method is same as method of finding minimum number of colour classes for vertex colouring in paper [13], see Proposition2.19. And Remark 2.20.

Proposition 4.1. Let $G$ be a graph of $k$ edges $e_1, e_2, e_3, ..., e_k$ and $C_1, C_2, C_3, ..., C_w$ are proper vertices of these $k$ edges, with maximum vertes number equal $n$, suppose all edges partition into $m$ colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, using the following method: In first step we try to find $q_0$ colour classes, which is maximum number of colour classes each has appearance equal $w$ at $v_1, v_2, v_3, ..., v_w$ vertices, such that each colour class consists only of common edges. In second step we try to find $q_1$ colour classes, which is maximum number of colour classes each has appearance equal $w-1$ at $v_1, v_2, v_3, ..., v_w$ vertices, such that each colour class consists only of common edges. In third step we try to find $q_2$ colour classes, which is maximum number of colour classes each has appearance equal $w-2$ at $v_1, v_2, v_3, ..., v_w$ vertices, such that each colour class consists only of common edges. We continue in same process, at last step we try to find $q_s$ colour classes, which is maximum number of colour classes each has appearance equal $w-s$ at $v_1, v_2, v_3, ..., v_w$ vertices, such that it is impossible to be $q_s + 1$ colour classes, and such that each colour class consists only of common edges. After last step if all common edges partition into colour classes $m$, where $m = q_0 + q_1 + q_2 + ... + q_s$, then $m$ is minimum number of colour classes.



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Proof: Suppose using only step I we partition all common edges into $m$ colour classes and neglect non common edges, due Proposition2.21, this means $m = q_0$. Using Proposition 2.20. We have $\sum_{i=1}^{m} t_i = \sum_{l=1}^{k_1} a_l + \sum_{l=1}^{k_2} b_l = \sum_{j=1}^{w} d(v_j) \le wn$, and since $n \le m$, we can write $\sum_{i=1}^{m} t_i = wq_0 = wm = \sum_{j=1}^{w} d(v_j) \le wn$, it a contradiction if $m < n$, then we have $n = m$, therefore $m$ is minimum number of colour classes. This means the result holds using only step I.

Suppose using only step I and step II we partition all common edges into $m$ colour classes and neglect non common edges, this means $m = q_0 + q_1$. Using Proposition 3.16. We can write $\sum_{i=1}^{m} t_i = wq_0 + (w-1)q_1 + r = \sum_{j=1}^{w} d(v_j) \le wn$, where $0 \le r$, and $r$ is number of non-common edges, from Proposition 2.21. after addition of these $r$ non-common edges, number of colour classes remain $m = q_0 + q_1$. Suppose all common edges partition into $m-1$ colour classes in state of $m$ colour classes, where $m-1 = q_0 + q_1 - 1$, then either there is $q_0 + x_0$ colour classes has appearance equal $w$ or there is there is $q_1 + x_1$ colour classes has appearance equal $w-1$, both $x_0, x_1$ are integers $0 \le x_0$, $0 \le x_1$ not both of them equal zero. Then there at least $q_0 + 1$ colour classes each has appearance equal $w$ at $v_1, v_2, v_3, ..., v_w$ vertices, or there at least $q_1 + 1$ colour classes each has appearance equal $w-1$ at $v_1, v_2, v_3, ..., v_w$ vertices, it is a contradiction if all common edges partition into $m-1$ colour classes. Then $m$ is minimum number of colour classes. This means the result holds using only step I and step II.

Suppose using only step I, stepII and step III we partition all common edges into $m$ colour classes and neglect non common edges, this means $m = q_0 + q_1 + q_2$. Using


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المجلة الدولية للبحوث العلمية
Vol. (3), No. (12) December 2024 الإصدار (3)، العدد (12)

Proposition 2.20. We can write $\sum_{i=1}^{m} t_i = wq_0 + (w-1)q_1 + (w-2)q_2 + r = \sum_{j=1}^{w} d(v_j) \leq wn,$ where $0 \leq r$, and $r$ is number of non-common edges. From Proposition 2.21. after addition of these $r$ non-common edges, number of colour classes remain $m = q_0 + q_1 + q_2$. Suppose all common edges partition into $m-1$ colour classes in state of $m$ colour classes, where $m-1 = q_0 + q_1 + q_2 - 1,$ then either there is $q_0 + x_0$ colour classes has appearance equal $w$ or there is there is $q_1 + x_1$ colour classes has appearance equal $w-1,$ or there is $q_2 + x_2$ colour classes has appearance equal $w-2,$ all of $x_0, x_1, x_2$ are integers $0 \leq x_0,$ $0 \leq x_1,$ $0 \leq x_2,$ not all of them equal zero. Then there at least $q_0 + 1$ colour classes each has appearance equal $w$ at $v_1, v_2, v_3, ..., v_w$ vertices, or there at least $q_1 + 1$ colour classes each has appearance equal $w-1$ at $v_1, v_2, v_3, ..., v_w$ vertices, or there at least $q_2 + 1$ colour classes each has appearance equal $w-2$ at $v_1, v_2, v_3, ..., v_w$ vertices, it is a contradiction if all common faces partition into $m-1$ colour classes. Then $m$ is minimum number of colour classes. This means the result holds using only step I, step II and step III. Continue in this process if $m = q_0 + q_1 + q_2 + ... + q_s,$ then $m$ is minimum number of colour classes. This means the result holds using all steps.

Remark 4.2. In this paper and coming papers we call the method in Proposition 4.1. method of finding minimum number of colour classes for edge colouring.

## V. How to Maximize Minimum Number of Colour Classes

Families of disjoint sets colouring technique, for edge colouring, as general technique include also graph of multiple edges. In this section, we explain how to find different values of minimum colour classes, for edge colouring, when maximum degree of vertex is fixed.

Remark 5.1. Let $G$ be a graph of $k$ edges $e_1, e_2, e_3, \ldots, e_k$ and $v_1, v_2, v_3, \ldots, v_w$ are proper vertices of these $k$ edges (standard of partition), with maximum degree of vertex equal $n,$ suppose these $k$ edges partitioned into minimum colour classes equal $m,$ from Proposition 3.4. Then $m$ is one value of $n, n+1, n+2, \ldots, 2n-1.$

In the following three examples we can substitute $n$ and $w$ by different values and sketch infinite number of graphs. In Examples 5.3. If $2w = n+2,$ and in Example 5.4. If $3w = n+3$ respectively, any graph is multiple graphs.

Example 5.2. Let $G$ be a graph of $k$ edges $e_1, e_2, e_3, \ldots, e_k$ and $v_1, v_2, v_3, \ldots, v_w$ are proper vertices of these $k$ edges, with maximum degree of vertex equal $n,$ suppose these $k$ edges partitioned into minimum colour classes equal $n+1,$ suppose $w$ is odd, and each colour class appears $w-1$ times at $v_1, v_2, v_3, \ldots, v_w.$ Using Proposition 2.10. We can write $(w-1)(n+1) = \sum_{j=1}^{w} d(v_j).$ since maximum value of $\sum_{j=1}^{w} d(v_j)$ equal $wn,$ then we can write $(w-1)(n+1) \le nw,$ and we have the condition $w \le n+1.$

Example 5.3. Let $G$ be a graph of $k$ edges $e_1, e_2, e_3, \ldots, e_k$ and $v_1, v_2, v_3, \ldots, v_w$ are proper vertices of these $k$ edges, with maximum degree of vertex equal $n,$ suppose these $k$ edges partitioned into minimum colour classes equal $n+2,$ suppose $w$ is odd, and each colour class appears $w-1$ times at $v_1, v_2, v_3, \ldots, v_w.$ Using Proposition 2.10. We can write $(w-1)(n+2) = \sum_{j=1}^{w} d(v_j).$ since maximum value of $\sum_{j=1}^{w} d(v_j)$ equal $wn,$ then we can write $(w-1)(n+2) \le nw,$ and we have the condition $2w \le n+2.$

Example 5.4. Let $G$ be a graph of $k$ edges $e_1, e_2, e_3, \ldots, e_k$ and $v_1, v_2, v_3, \ldots, v_w$ are proper vertices of these $k$ edges, with maximum degree of vertex equal $n,$ suppose these $k$ edges partitioned into minimum colour classes equal $n+3,$ suppose $w$ is odd, and


International Journal for Scientific Research, London Vol (3), No (12), 2024
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each colour class appear $w-1$ times at $v_1, v_2, v_3, ..., v_w$. Using Proposition 2.10. We can write $(w-1)(n+3) = \sum_{j=1}^{w} d(v_j)$. since maximum value of $\sum_{j=1}^{w} d(v_j)$ equal $wn$, then we can write $(w-1)(n+3) \le nw$, and we have the condition $3w \le n+3$.

**The following proposition is generalization of above three examples:**

Proposition 5.5. Let $G$ be a graph of $k$ edges $e_1, e_2, e_3, ..., e_k$ and $v_1, v_2, v_3, ..., v_w$ are proper vertices of these $k$ edges, with maximum degree of vertex equal $n$, suppose these $k$ edges partitioned into minimum colour classes equal $n+1$, suppose $w$ is odd, and each colour class appears $w-1$ times or all colour classes (except one colour class) appears $w-1$ times at $v_1, v_2, v_3, ..., v_w$. Then $xw \le n+x$ is condition if these $k$ edges partitioned into minimum number colour classes equal $n+x$. where $x \le n-1$.

Proof: Suppose these $k$ edges partitioned into minimum colour classes equal $m = n+x$, from Proposition 3.5. We have $n+x = m \le 2n-1$, and $x \le n-1$, since each colour class appears $w-1$ times at $v_1, v_2, v_3, ..., v_w$. Using Proposition 2.10. We can write $(w-1)(n+x) \le \sum_{j=1}^{w} d(v_j)$. since maximum value of $\sum_{j=1}^{w} d(v_j)$ equal $wn$, then we have $(w-1)(n+x) \le nw$, and $xw \le n+x$.

The proof of the following Proposition is direct from definition of appearance of an edge. Proposition 5.6. Let $G$ be a graph of $k$ common edges $e_1, e_2, e_3, ..., e_k$ and $v_1, v_2, v_3, ..., v_w$ are proper vertices of these $k$ edges, if $a_i$ is appearance of the edge $e_i$, of where $1 \le i \le k$, then $2 \le a_i \le w$.



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المجلة الدولية للبحوث العلمية
Vol. (3), No. (12) December 2024 الإصدار (3)، العدد (12)

Proposition 5.7. Let $G$ be a graph of $k$ common edges $e_1, e_2, e_3, ..., e_k$ and $v_1, v_2, v_3, ..., v_w$ are proper vertices of these $k$ edges, if $w(w-1)+2 \leq \sum_{j=1}^{w} d(v_j)$ then $G$ has a multiple edge.

Proof: If $G$ is complete graph, then $w(w-1) = \sum_{j=1}^{w} d(v_j)$, but $w(w-1)+2 \leq \sum_{j=1}^{w} d(v_j)$, then $G$ has a multiple edge.

In view of Proposition 5.7. We introduce the following proposition without proof.

Proposition 5.8.Let $G$ be a graph of $k$ common edges $e_1, e_2, e_3, ..., e_k$ and $v_1, v_2, v_3, ..., v_w$ are proper vertices of these $k$ edges, if $G^*$ is subgraph of $G$, where $G^*$ be a graph of $k^*$ common edges and $v^*{}_1, v^*{}_2, v^*{}_3, ..., v^*{}_{w^*}$ be $w^*$ proper vertices of these $k^*$ edges, and $w^* < w$, $k^* < k$, $1 \leq j \leq w^*$, if $w^*(w^*-1)+2 \leq \sum_{j=1}^{w^*} d(v^*{}_j)$, then $G$ has multiple edges.

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